THE SOUNDNESS OF FIRST AND SECOND ORDER S5 MODAL LOGIC[1]

A. BAYART

I: *Semantic definitions*

0. To formulate a semantic theory of modal logic it is not sufficient to define for example, the necessary as that which is true in every model and the possible as that which is true in some model. These definitions would do no more than introduce the notions of 'necessary' and 'possible' in the metalanguage. A semantics of modal logic demands that we assume an object language containing modal symbols and that we define under what conditions to attribute the values 'true' or 'false' to the formulae of this object language.

One can then very easily define the validity and satisfiability of formulae in this language and shew the soundness of such and such a deductive system, this soundness consisting in the fact that all derivable formulae in the considered systems are valid.

It is a theory of this kind which we propose to develop in the present article, inspired in us by the Leibnizian definition of necessity as truth in all possible worlds.

It is not, in our opinion, the task of the logician to examine the value of this Leibnizian metaphysics. We can confine ourselves to shewing that if one takes this metaphysics one can formulate for the modal logic S5 a semantical theory analogous to the formal semantic theories of non-modal logic.

The modal semantic theory leads us to consider predicates of a special kind. These are predicates whose extension varies in one world or another, and we give them the name of 'intensional predicates'.

1. Letting A and B be two non-empty sets, not having any common elements, call A the 'set of individuals' and B the 'set of worlds'. We say that these sets A and B constitute a universe U, For each natural number n we mean by 'n-place intensional predicate' a function of n+1 arguments, taking the value 'true' or 'false' having a world as its first argument and for n ≠ 0, having n individuals as its last n arguments.

Letting a and b be the cardinal numbers of A and B, for any natural number n, there are c = 2exp (b(a exp n)) n-place intensional predicates.

2. We take a language L. For the moment we confine ourselves to considering a language without axioms or rules of deduction. This language contains a denumerable infinity of individual variables, and for each natural number n, a denumerable infinity of n-place predicate variables. It does not contain constants for individuals or predicates.

In the following exposition the different types of variables will be designated indifferently by the small letters which play the rõle of syntactical variables. Certain small letters can also designate other expressions than variables. We indicate each time in the context what sort of expressions are designated by the syntactical variables. These syntactical variables may be followed by numbers, and we write e.g., x0, x1, x2,..., xn.

We shall adopt the Polish notation. The language L contains the symbols N, K, A, C, and E for negation, conjunction, disjunction, implication and equivalence, the symbols P and S (in lieu of pi and sigma) for the universal and existential quantifiers, and the symbols L and M for necessity and possibility. We introduce these capital letters not only in the object language but also in the metalanguage where they are combined with syntactical variables to form complex syntactical expressions.

Formation rules are as follows:

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[1]Translation by M.J. Cresswell of 'La correction de la logique modale du premier et second ordre S5', *Logique et Analyse*, 1, 1958, pp. 28–44. This version of the translation retains Bayart's own terminology and notation. [..] indicates editorial comment, especially cases where there appear to be typos, though I don't guarantee to have marked all cases of apparent typos. (All footnotes are my comment on the translation.)

a.) a 0-place predicate variable is a proposition
b.) an n-place predicate variable followed by n individual variables is a proposition.
c.) If p is a proposition Np is a proposition.
d.) If p and q are propositions then Kpq is a proposition
e.) If p and q are propositions then Apq is a proposition
f.) If p and q are propositions then Cpq is a proposition
g.) If p and q are propositions then Epq is a proposition
h.) If p is a proposition and v is a variable then Pvp is a proposition
i.) If p is a proposition and v is a variable then Svp is a proposition
j.) If p is a proposition Lp is a proposition.
k.) If p is a proposition Mp is a proposition.
l.) There are no other propositions

We have thus a pure modal second order language.

3. Let there be a universe U composed of the set A of individuals and B of worlds. We agree that the variables for individuals of the language L can take as the values the individuals of the set A and that for each natural number n the variables for n-place predicates take as their values the intensional predicates defined from the universe U.

We take a universe U, a world M of this universe and a system of values S relative to this universe. We then define the notions 'true for universe U, the world M and the value system S', and 'false for universe U, the world M and the value system S'. let f be a proposition of language L.

If f is a variable p for 0-place predicates, if P is the 0-place intensional predicate given as the value of p, f will be true or false for UMS according as P takes the value 'true' or 'false' when it receives M as argument.

If f is of the form bx1...xn, where b is an n-place predicate variable (n ≠ 0) and where x1,...,xn are individual variables if B, X1,..., Xn are respectively the n-place intensional predicate and the individuals given as values of b, x1,..., xn, f will be true or false for UMS according as B takes the values 'true' or 'false' when it receives M, X1, ..., Xn as arguments in that order.

If f has the form Np, where p is a proposition, f will be true for UMS if p is false for UMS, and false for UMS if p is true for UMS.

If f has the form Kpq, where p and q are propositions, f will be true for UMS if p and q are true for UMS, and false for UMS if not.

If f has the form Apq, where p and q are propositions, f will be true for UMS if p is true for UMS or if q is true for UMS, and false for UMS if not.

If f has the form Cpq, where p and q are propositions, f will be true for UMS if p is false for UMS or if q is true for UMS, and false for UMS if not.

If f has the form Epq, where p and q are propositions, f will be true for UMS if p and q are true for UMS or if p and q are false for UMS, and false for UMS if not.

If f has the form Pvp, where p is a proposition and v a variable for individuals or predicates, f will be true for UMS if, for each system S′ relative to U which gives to all the variables other than v the same values as S, p is true for UMS′. Otherwise f is false for UMS.

If f has the form Svp, where p is a proposition and v a variable for individuals or predicates, f will be true for UMS if there is a system S′ relative to U which gives to all the variables other than v the same values as S, and p is true for UMS′. Otherwise f is false for UMS.

If f has the form Lp, where p is a proposition, f will be true for UMS if for every world M′ of the universe U, p is true for UM′S, and otherwise f will be false for UMS.

If f has the form Mp, where p is a proposition, f will be true for UMS if there is a world M′ of the universe U such that p is true for UM′S, and otherwise f will be false for UMS.

4. We take a universe U and a world M of this universe. We define for propositions of the language L the notions of 'valid in UM' and 'satisfiable in UM'. Let f be a proposition of L.

The proposition f will be valid in UM if and only if, for each system S of values relative to U, f is true for UMS.

The proposition f will be satisfiable in UM iff there is a system S of values relative to U such that f is true for UMS.

The proposition f will be valid in U iff it is valid in every UM (for every world M).

The proposition f will be satisfiable in U iff there is some world M such that f is satisfiable in UM.

We define for the language L the notions 'valid' and 'satisfiable'.

The proposition f will be valid iff it is valid in all universes.

The proposition f will be satisfiable iff it is satisfiable in some universe.

We transform the language L into a system of deduction D by giving axioms and rules of deduction. D will be sound if one can only prove valid formulae. D will be complete if one can prove any valid formula of the language L.

II *Auxiliary language*

5. From the expressions of the language L we form an auxiliary language L′ by introducing the letter Z. This symbol will play the same role as the traditional symbol lambda.

The expressions of L′ will play a syntactical role and so appear in the metalanguage. They designate certain expressions of L which will be called the resultants of corresponding expressions of L′.

In the exposition which follows we continue to use small letters to indicate syntactical variables and combine them with the logical constants N, K, A, C, E, P, S, L, M and the operator Z to form complex syntactical expressions.

The letter Z followed by a finite number n ($n \neq 0$) of individual variables is an n-place individual abstractor.

The letter Z followed by a 0-place predicate variable is a propositional abstractor.

The letter Z followed by an n-place predicate variable ($n \neq 0$) is an n-place predicate abstractor.

An n-place individual abstractor followed by a proposition is an n-place parapredicate (or an n-place individual abstract.)

A propositional abstractor followed by a proposition is a propositional abstract.

A predicate abstractor followed by a proposition is a predicate abstract.

An n-place parapredicate followed by n individual variables (not necessarily distinct) is a primary, simple, paraproposition.

The expression obtained by substituting in any way whatever in a proposition n-adic parapredicates for n-adic predicate variables is a primary complex paraproposition.

One sees that in a complex primary paraproposition the parapredicates with n0, n1, n2, ... variables for individuals are followed respectively by the n0, n1, n2, ... variables for individuals which follow the variables for predicates of n0, n1, n2, ... places in the original proposition. These parapredicates will thus form, with variables for individuals, simple primary parapropositions.

A propositional abstract followed by a proposition is a propositional secondary paraproposition. A predicate abstract where the abstractor is an n-place predicate variable ($n \neq 0$) followed by an n-place parapredicate is a predicate secondary paraproposition.

(Note: In the exposition which follows we introduce parentheses into expressions of the auxiliary language for ease of reading.)

6. In an abstract the free variables are the variables which occur free in the proposition which follows the abstractor, other than the variables in the abstractor.

In an abstract the bound variables are the variables of the abstractor, the bound variables which follow the abstractor and the free variables of this proposition which occur equally in the abstractor. One says of these last variables that they are bound by the abstractor. In particular in a parapredicate Zx1...xn(p), where p is a proposition, if the variable xi (i = 1,2,...n) appears free in p it is said to be bound by the i-th variable of the abstractor.

The resultant of a simple primary paraproposition Zx1...xn(p)a1...an is the proposition p′ which is obtained by simultaneously substituting in the proposition p individual variables a1,...,an for the individual variables x1,...,xn wherever they occur free in p.

Substitution is simultaneous if at each place in p where a variable occurs bound by the abstractor one makes one and only one substitution.

The resultant of a complex primary paraproposition is a proposition p′ which one obtains by replacing in p each simple primary paraproposition by its resultant.

The resultant of a propositional paraproposition Zq(p)j is the proposition p′j which is obtained by substituting in the proposition p the proposition j for the propositional variable q wherever it occurs free in p.

The intermediate resultant of a predicate paraproposition Zb(p)k, where k is a parapredicate of the same number of places as the variable p, is the complex primary paraproposition obtained by substituting in the proposition p the parapredicate k for the variable b wherever the latter occurs free in p.

The final resultant, or more briefly the resultant, of a predicate paraproposition is the resultant of the intermediate resultant.

7. A simple primary paraproposition Zx1...xn(p)a1...an is well-formed if for every i (i = 1, 2, ...n) the variable xi does not occur free in p in the scope of a quantifier Pai or Sai. A complex primary paraproposition is well-formed if all its simple primary parapropositions are well-formed.

A propositional paraproposition Zq(p)j is well-formed if the variable q does not occur free in p within the scope of a quantifier Pv or Sv where v is any variable which occurs free in j.

A predicate paraproposition Zb(p)k is well-formed if

1. The variable b does not occur free in p in the scope of a quantifier Pv or Sv where v is any variable which occurs free in the parapredicate k

and if also

2. The intermediate resultant of Zk(p)k is well-formed.

III: *Sematic properties of parapropositions*

8. Definition. We give a recursive definition of the notion 'modalised proposition'.[2]

1. Propositions of the form Lp and Mp are modalised
2. If p is a modalised proposition then Np is a modalised proposition.
3. If p and q are modalised propositions then Kpq Apq Cpq and Epq are modalised propositions.[3]
4. If p is a modalised proposition and if v is a variable then Pvp and Svp are modalised propositions.
5. There are no other modalised propositions than those defined by rules 1-4.

9. Definition. The value of a parapredicate Zx1...xn(p) for a universe U and a value system S relative to U is the n-place intensional predicate which, for every world M and any series of individuals A1, ..., An, takes the value 'true' or 'false' according as the proposition p takes the value 'true' or 'false' for UMS′, where S′ is a value system which assigns individuals A1,..., An as values to the individual variables x1,..., xn respectively and which gives to all other variables the same values as S.

10. *Theorem I.* Consider a universe U, two worlds M and M′ of U and any value system relative to U. If p is a modalised proposition then p has the same value for UMS and UM′S.

Proof by induction on the definition of modalised formulae.

11. *Theorem II.* Let p be a proposition containing only v1,..., vn as free variables. Consider any universe U, a world M of U and two value systems S and S′ relative to U which do not differ in the values assigned to v1,..., vn. Then p takes the same value for UMS and UMS′. In particular if p is a closed proposition (i.e., does not contain free variables) then for any two value systems S and S′ relative to U, p takes the same value for UMS and UMS′.[4]

Proof by induction on the construction of p.

12. *Theorem III.* Let k be a parapredicate Zx1...xn(p) which contains only the variables v1,..., vn free. Take any universe U, any world M of U and any two value systems S and S′ relative to U which do not differ in the values given to the variables v1,..., vn. Then k takes the same value for US and for US′. In particular if k is a closed parapredicate (i.e., does not contain free variables) then for any universe U, and any two value systems S and S′ relative to U, k takes the same value for US and US′.

In the proof we rely on the definition of a parapredicate and on the result of theorem II.

13. *Theorem IV.* For any universe U, and world M of U and any system S of values relative to U, if Zx1...xn(p)a1...an is a well-formed simple primary paraproposition the value for UMS of the resultant p′ of this proposition is the same as the value of p for UMS′, where S′ is a value system which gives to the individual variables x1,...,xn the individuals A1,..., An respectively, being the same individuals as assigned by S to the variables a1,..., an respectively, and which gives all other variables the same values as S does.

Proof by induction on the construction of p.

14. *Theorem V.* For any universe U, any world M of U, and any value system S relative to U, if Zx1...xn(p)a1...an is a well-formed simple primary paraproposition, and if P is the n-place intensional predicate which is the value for US of the parapredicate Zx1...xn(p), the value for UMS of the resultant

[2] I have translated Bayart's «proposition couverte» here not as 'closed proposition' but as 'modalised proposition', or strictly speaking 'completely modalised proposition', since this is in accordance with standard usage in modal logic.

[3] I have corrected what appears to be a typo. Bayart's Opq should clearly be Cpq.

[4] Bayart just has 'for two value systems' but 'any' seems necessary to give the correct sense.

p′ of this paraproposition is that taken by the predicate P when it is given as arguments the world M and the individuals A1,..., An these last being in this order the values given by S to the variables a1,..., an.

The proof relies on the definition of a parapredicate and the result of theorem IV.

15. *Theorem VI.* For any universe U, any world M of U, and any value system S relative to U, if Zy(p)q is a well-formed propositional paraproposition, the value for UMS of the resultant p′ of this paraproposition is the same as that of the proposition p for UMS′ where S′ is the value system which gives the propositional variable y the value 'true' or 'false' according as the proposition q has the value 'true' or 'false' for UMS, and which gives all the other variables the same value as S.

Proof by induction on the construction of p.

16. *Theorem VII.* For any universe U, any world M of U, and any value system S relative to U, if Zb(p)k is a well-formed predicate paraproposition where b is an n-place predicate variables and k is an n-place parapredicate the value for UMS of the final resultant p″ of this paraproposition is the same as the value of the proposition p for UMS′ where S′ is the value system which assigns to the variable b the value which the parapredicate k takes for US and which gives all the other variables the same values as S.

Proof by induction on the construction of p.

IV *Soundness of the second order S5.*

17. We formulate S5 by means of Gentzen systems. A sequent comprises first a finite series,[5] possibly empty, of propositions of the language L. This series is called the 'antecedent', second, the symbol I, third a finite series, possibly empty, of propositions of the language L. This series is called the 'consequent'.

The system S5 comprises an axiom schema and twenty eight rules of deduction divided into four groups: structural rules, propositional rules, quantificational rules and modal rules. The rules permit the passage from one or two sequents called premises to another sequent called the conclusion.

To formulate the axiom schema and the rules of deduction we use a metalanguage containing, among other things, the expressions which we used in formulating the theory of parapropositions. We will also include the symbol I in the metalanguage.

In particular, in the present section IV, the letters p and q will designate propositions and the letters ä, ä′ and ë, ë′ will designate series of propositions. In an expression of the form Pvp or Svp, the letter v designates an individual variable, a propositional variable or a variable for a predicate of n-places. (n > 0) [(n = o)] In an expression of the form Zv(p)a the letter a designates an individual variable, a proposition or an n-place parapredicate, according as v designates an individual variable, a propositional variable or an n-place predicate variable. An expression of the form Zv(p)a designates a paraproposition, but it is understood that it is not the parapropositions but the resultants of the parapropositions which figure in deductions.

In the antecedent expressions designating propositions or series of propositions are followed by commas, and in the consequent these expressions are preceded by commas.

18. We define the notions 'true' and 'false' for Gentzen sequents relative to a universe U, a world M and a system of values S.

A sequent ä,I,ë is true for UMS if ä contains a proposition false for UMS or if ë contains a proposition true for UMS. Otherwise the sequent ä,I,ë is false for UMS.

Following from this definition we can, as we have done in paragraph 4 for propositions, define for sequents the notions 'valid for UM', valid for U', 'valid', 'satisfiable for UM', 'satisfiable for U',

[5] I have retained Bayart's word «series» here, though perhaps 'sequence' would be more appropriate. I have translated Bayart's word «sequence» in the context of Gentzen systems as 'sequent'.

‘satisfiable’.

We shew that the system S5 is sound in the sense that all deductions are valid sequents. We shew, in particular, that the axioms are valid, and that the rules of deduction are such that, if the premises are valid, the conclusion is valid. It is convenient here to recall that ‘valid’ is synonymous with ‘true for every universe U, every world M of this universe, and every system of values S relative to this universe’.

We shew the soundness of the axiom (or, what comes to the same thing, the axiom schema) and the rules of deduction as we present them.

19. The axiomatic schema (which we label ‘Ax’) is the following

p,I,p

The axioms which are instances of this schema are obviously valid. If p designates a true proposition, ë contains a true proposition and if p designates a false proposition then ä contains a false proposition.[6]

There are seven structural rules; to be precise, addition, permutation and contraction in the antecedent (designated, respectively, by ‘ADI, ‘PEI’, and ‘COI’), addition, permutation and contraction in the consequent (designated, respectively, by ‘IAD, ‘IPE’, and ‘ICO’) and cut (designated by ‘Cut’).

The rules are as follows:

ADI	ä,I,ë —————— p,ä,I,ë	
	ä,I,ë —————— ä,I,ë,p	IAD
PEI	ä,p,q,ä′,I,ë —————— ä,q,p,ä′,I,ë	
	ä,I,ë,p,q,ë′ —————— ä,I,ë,q,p,ë′	IPE
COI	p,p,ä,I,ë —————— p,ä,I,ë	
	ä,I,ë,p,p —————— ä,I,ë,p	ICO
	ä,I,ë,p p,ä,I,ë —————————— ä,I,ë	Cut

The soundness of the rules with one premise is obvious. The proof of the soundness of Cut is as follows. Assume a universe U, a world M in this universe, and a system S of values in this universe. By hypothesis the two premises are true for UMS. So ä [a] will contain a proposition false for UMS or ë [e] will contain a proposition true for UMS, for otherwise p would have to be true for the first premise to be true, and p would have to be false for the second premise to be true. It follows that the conclusion is true for UMS.

20. There are ten propositional rules; to be precise, the introduction of N, K, A, C and E in the antecedent (designated respectively by ‘NI’, ‘KI’, ‘CI’ and ‘EI’, and the introduction of N, K, A, C and E in the

[6] Bayart has e in place of my ä and ë. The latter seem to make more sense here. At any rate, the intent is clear. If p is true the consequent contains (is) a true proposition, and if p is false the antecedent contains (is) a false proposition.

consequent (designated respectively by ‘IN’, ‘IK’, ‘IC’ and ‘IE’.

The rules are as follows:

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       ä,I,ë,p                                 p,ä,I,ë
NI ____________________            ____________________ IN
       Np,ä,I,ë                                ä,I,ë,Np


       p,q,ä,I,ë                               ä,I,ë,p        ä,I,ë,q
KI ____________________            ______________________________ IK
       Kpq,ä,I,ë                                      ä,I,ë,Kpq

       p,ä,I,ë        q,ä,I,ë                        ä,I,ë,p,q
AI ____________________________        ____________________ IA
              Apq,ä,I,ë                              ä,I,ë,Apq

       ä,I,ë,p        q,ä,I,ë                        p,ä,I,ë,q
CI ____________________________        ____________________ IC
              Cpq,ä,I,ë                              ä,I,ë,Cpq

       ä,I,ë,p,q      p,q,ä,I,ë            p,ä,I,ë,q      q,ä,I,ë,p
EI ____________________________   ________________________________ IE
              Epq,ä,I,ë                              ä,I,ë,Epq
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We prove the soundness of EI and IE

For EI: Consider a universe U, and world M and a system of values S relative to this universe. If ä [M] contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS. If ä does not contain a proposition false for UMS and if ë does not contain a proposition true for UMS, then, since by hypothesis the first premise is true for UMS it is necessary that one of the propositions p or q is true for UMS, and, since by hypothesis the second premise is true for UMS it is necessary that one of the propositions p or q is false for UMS. If one of the two propositions p and q is true and the other is false, Epq will be false for UMS, and so the conclusion will be true for UMS.

For IE: Consider a universe U, and world M and a system of values S relative to this universe. If ä contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS. If ä does not contain a proposition false for UMS or if ë does not contain a proposition true for UMS, two cases are possible: If p is true for UMS then, since by hypothesis the first premise is true for UMS it is necessary that q will be true for UMS. If p is false for UMS then, since by hypothesis the second premise is true for UMS it is necessary that q is false for UMS; if p and q are true for UMS or if p and q are false for QMS. Epq is true for UMS, and so the conclusion will be true for UMS.

21. There are four rules of quantification; to be precise the introduction of P and S in the antecedent (designated, respectively, by ‘PI, ‘SI’) and the introduction of P and S in the consequent (designated, respectively, by ‘IP’, ‘IS’).

The rules are as follows:

Zv(p)a,ä,I,ë ä,I,ë,p

PI ________________ ________________ IP

Pvp,ä,I,ë ä,I,ë,Pvp

p,ä,I,ë ä,I,ë,Zv(p)a

SI ________________ ________________ IS

Svp,ä,I,ë ä,I,ë,Svp

Restriction (1): In the rules PI and IS, Zv(p)a must be a well-formed paraproposition.

Restriction (2): In the rules IP and SI the variable v cannot occur free in the propositions of ä or of ë.

We prove the soundness of PI and of IP

For PI: Consider a universe U, and world M and a system of values S relative to this universe. If ä contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS. If ä does not contain a proposition false for UMS and if ë does not contain a proposition true for UMS, then, since by hypothesis the first premise is true for UMS, Zv(p)a will be false for UMS. It follows by virtue of theorems IV, VI or VII that there is a system of values S′, which gives the value to v that S gives to a, and which gives all other variables the same value that S gives, and is such that p is false for UMS′. There is thus a system of values S′, which is no different from S except for the value given to v, and is such that p is false for UMS′. So Pvp is false for UMS, and so the conclusion will be true for UMS.

For IP: Consider a universe U, and world M and a system of values S relative to this universe. If ä contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS. If ä does not contain a proposition false for UMS and if ë does not contain a proposition true for UMS, since the propositions in ä and in ë do not contain free v, it follows, in virtue of theorem II that, for every system S′ which gives to all variables except v the same values as S, ä will not contain propositions which are false for UMS′, and that ë will not contain propositions which are true for UMS′. Now, by hypothesis, for all these systems S′, the premises are true for UMS′. Thus, for all these systems S′, p will be true for UMS′. Thus Pvp is true for UMS, and so the conclusion will be true for UMS.

22. There are four modal rules; to be precise the introduction of L and M in the antecedent (designated, respectively, by ‘LI, ‘MI’) and the introduction of L and M in the consequent (designated, respectively, by ‘IL’, ‘IM’).

The rules are as follows:

ä,I,ë,p p,ä,I,ë

LI ________________ ________________ IL

Lp,ä,I,ë ä,I,ë,Lp

p,ä,I,ë ä,I,ë,p

MI ________________ ________________ IM

Mp,ä,I,ë ä,I,ë,Mp

Restriction (3): In the rules IL and MI the propositions of ä and ë must be fully modalised.

We prove the soundness of LI and of IL

For LI: Consider a universe U, and world M and a system of values S relative to this universe. If ä contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS.

If ä does not contain a proposition false for UMS and if ë does not contain a proposition true for UMS, then, since by hypothesis the first premise is true for UMS, it is necessary that p will be false for UMS. So Lp is false for UMS and the conclusion will be true for UMS.

For IL: Consider a universe U, and world M and a system of values S relative to this universe. If ä contains a proposition false for UMS, or if ë contains a proposition true for UMS the conclusion is true for UMS. If ä does not contain a proposition false for UMS and if ë does not contain a proposition true for UMS, since the propositions in ä and in ë are fully modalised, it follows, in virtue of theorem I that, for every world M′, ä will not contain propositions which are false for UM′S, and that ë will not contains propositions which are true for UM′S. Now, by hypothesis, for all these worlds M′, the premises are true for UM′S. Thus, for all worlds M′, p will be true for UM′S. Thus Lp is true for UMS, and so the conclusion will be true for UMS.

V *First-order logic*

23. From the preceding one can easily extract the theory of first-order modal logic.

First-order modal logic contains a denumerable infinity of individual variables, and, for each natural number n, a denumerable infinity of variables for n-place predicates.

The formation rules are the same as for second-order logic, except that, in expressions of the form Pvp or Svp, v must be an individual variable.

24. The semantic definitions are the same as for second-order modal logic.

25. In first-order modal logic we need only consider abstractors containing just an individual variable, and so simple primary parapropositions of the form Zx(p)a, where x is an individual variable.

26. We only need theorems I, II and IV. In the last theorem we only need to consider parapropositions formed by abstractors containing a single individual variable.

27. The deduction rules are the same as in second-order logic but the scope of the quantification rules is automatically reduced, when we note that, in expressions of the form Pvp, Svp and Zv(p)a, v designates an individual variable, and excludes predicate variables.

The soundness of first-order modal logic can be proved in the same way as in second-order modal logic.

VI *Necessity and validity*

28. One might perhaps combine the notions of necessity and validity. One might then formulate the following semantic theory:

Instead of providing a universe consisting of a domain and a set of individuals one might simply give a domain D, i.e., a set of individuals. One then gives a set of extensional predicates. For each natural number n, an extensional predicate is a function of n arguments, these arguments being individuals, and able to take the values ‘true’ or ‘false’.

Individual variables can take individuals as values, and n-place predicate variables can take n-place extensional predicates as values. Propositional variables can take ‘true’ or ‘false’ as values.

29. Assume a domain D and a value-system S.

A propositional variable is true or false for D according as S gives it the value ‘true’ or ‘false’.

A proposition of the form bx1...xn, where b is an n-place predicate variable and x1,...,xn are individual variables, will be true for DS if the extensional predicate B given to b as value by S has the value ‘true’ when given as arguments the individuals X1,..., Xn being the values given in this order to x1,..., xn respectively; and bx1...xn will be false for DS if B has the value ‘false’ when given X1,..., Xn as arguments.

A proposition of the form Np is true for DS if p is false for DS, and otherwise false for DS.

A proposition of the form Kpq is true for DS if p and q are true for DS, and otherwise false for DS.

A proposition of the form Apq is true for DS if p is true for DS or q is true for DS, and otherwise false for DS.

A proposition of the form Cpq is true for DS if p is false for DS or q is true for DS, and otherwise false for DS.

A proposition of the form Epq is true for DS if p and q are both true for DS or if p and q are both false for DS, and otherwise false for DS.

A proposition of the form Pvp is true for DS if for every value system S′, which gives all variables except v the same values as S does, p is true for DS′, and otherwise it is false for DS.

A proposition of the form Svp is true for DS if there is a value system S′, which gives all variables except v the same values as S does, and p is true for DS′, and otherwise it is false for DS.

A proposition of the form Lp is true for DS if for every value system S′, p is true for DS′, and otherwise it is false for DS.

A proposition of the form Mp is true for DS if there is a value system S′, such that p is true for DS′, and otherwise it is false for DS.

30. A proposition is valid for D if, for every value system S it is true for DS.

A proposition is satisfiable for D if, there is a value system S such that it is true for DS.

A proposition is valid if it is valid for every domain D.

A proposition is satisfiable if, there is a domain D such that it is satisfiable for D.

We can turn our language L into a deductive system by giving axioms and deduction rules.

The deductive system is sound if one can only prove valid formulae.

The deductive system is complete if one can prove all valid formulae.

31. The semantic rules that we have just given make first-order S5 unsound, and equally in the second-order case.

In first-order modal S5, and a fortiori in second-order modal S5, we have the following deduction:

bx,I,bx
__________________ IN

I,bx,Nbx
__________________ IA

I,AbxNbx
__________________ IL [Bayart has IA]

I,LAbxNbx
__________________ IS

I,SyLAbxNby

The conclusion of this deduction is not valid in the semantics proposed in section VI. [VII]

Assume a domain D composed of two individuals 0 and 1. Let B be a one-place predicate such that B0 is true and B1 is false. Let S be a value system which gives the value B to b and 1 to x, whatever values it gives to the other variables of L. The proposition SyLAbxNby will be false for DS.

For let S″ be a value system which gives the value B to b, 1 to x and 0 to y, and where it does not matter what values it gives to the other variables of L. We have successively:

bx is false for DS″

by is true for DS″

Nby is false for DS″

AbxNby is false for DS″

For all systems S‴ LAbxNby is false for DS‴ [Bayart has LabxNby]

In particular, for every system S′ which gives all other variables the same values as S, LAbxNby is false for DS′. So SyLAbxNby is false for DS.

32. In second-order modal S5 we have the following deduction:

bx,I,bx
__________________ IN

I,bx,Nbx
__________________ IA

I,AbxNbx
__________________ IL

I,LAbxNbx
__________________ IS

I,ShLAbxNhx

The conclusion of this deduction is not satisfiable in the semantics proposed in section VII.

Assume a domain D and a value system S. Let X be an individual in the domain D. Let B be a one-place predicate such that BX [Bayart has Bx] is false. Let H be a one-place predicate such that HX is true

Then let S″ be a value system which gives the value X to x and B to b and H to h, whatever values it gives to the other variables of L. We have successively:

bx is false for DS″

hx is true for DS″

Nhx is false for DS″

AbxNhx is false for DS″

For all systems S‴ LAbxNhx is false for DS‴ [typo: Bayart has LAbxMhx]

In particular, for every system S′ which gives all other variables the same values as S, LAbxNhx is false for DS′. So ShLAbxNhx is false for DS.

The proof holds for every domain D and value-system S.

33. The problem with the sematic theory presented in section VI lies in the fact that it treats the symbols L and M as abbreviations for universal and existential closures. So that in expressions of the form SyLAbxNby or ShLAbxNhx the variables y and h are considered to be bound by L and M and not by the quantifiers Sy or Sh, as they are in modal logic. Modal logic does not treat L and M as abbreviations for universal or existential closure. In other words modal logic does not identify the notions of necessity and validity.

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A. BAYART. *(Brussels)*.