ARNOULD BAYART'S MODAL COMPLETENESS THEOREMS

Translated with an introduction and commentary by M.J. Cresswell

1. Historical introduction[1]

Contemporary modal logic originated in 1912 with an article by C.I.Lewis in *Mind*, and was developed by him in other articles, and most particularly in two books, Lewis 1918 and Lewis and Langford 1932. But until the late 1950s there was no adequate semantics for it which would allow a definition of validity to be compared with the various axiomatic systems. Axiomatic modal predicate logic appeared in Barcan 1946a. 1946b and 1947. The first attempt to provide a semantics for modal predicate logic occurs in Carnap 1946 and 1947, but it was not until 1958 and 1959 that the breakthrough came. In Bayart 1958 we have a definition of validity for first and second-order S5, and in Bayart 1959 and Kripke 1959 we have two quite different completeness proofs for modal predicate S5. Kripke's article in *The Journal of Symbolic Logic* became widely known, and Kripke developed his semantics to include other systems based on the relational semantics for propositional modal logic developed by such authors as Meredith and Prior 1956, Hintikka 1957, Kanger 1957 and others.[2]

Bayart's work discusses only S5, and there is no evidence that he was familiar with the relational semantics for other systems. There are however a number of respects in which his work deserves acknowledgement. What makes Bayart's work most significant is the fact that the later paper, Bayart 1959, is the first completeness proof for modal predicate logic based on the Henkin construction of maximal consistent sets (Henkin 1949), and indeed may be the earliest application of the Henkin method even to propositional modal logic.[3] Kripke's paper proves the completeness of first-order S5 by the method of Beth trees. Kripke hints in his paper at the extensions needed for other systems, but does not cover them in the 1959 paper, and indeed Kripke's later completeness proofs, with the exception of his proof for first-order intuitionistic logic with its connection to S4 (Kripke 1965), mainly concerned propositional logic.[4]

Carnap's work predates both Bayart's and Kripke's by more than a decade, and like Bayart's articles and like Kripke 1959 Carnap dealt only with S5. But Carnap attempted to derive necessity from validity, and it is at least controversial whether such a procedure can work. The important insight, often credited to Kripke 1963a, is that if we think of the necessary as that which it is true in all possible worlds then it does not matter what the worlds are. They can, as earlier theorists, including Kripke 1959, had often supposed, be models or assignments or some such linguistic entity, but they do not have to be. The opening sentence of Bayart 1958 states that 'it is not sufficient to define for example, the necessary as that

[1]Much of the information on the state of logic in Belgium at the time Bayart was working has been obtained with the assistance of Jacques Riche, who is preparing an account of the history of logic in Belgium. That article will be provided on line, and, hopefully, in printed form, and will provide an essential complement to the story told here. I was able, with Riche's assistance, to interview Paul Gochet and Hubert Hubien, who supplied us with helpful information. Other help is acknowledged in the acknowledgements section at the end of this article.

[2]I have restricted myself here to mentioning work which appeared before or at the same time as Bayart's work. In some cases, e.g., Montague 1960, authors have claimed that the work published was available much earlier, so this article cannot be held to making definitive historical claims about priority. I have been assured that the relevant issues of *Logique et Analyse* did appear in 1958 and 1959. There is no indication of when they were accepted for publication.

[3]Certainly Kaplan 1966, p. 121f gives the impression that no Henkin completeness proof has been produced for modal systems. Kaplan of course is thinking of systems with a relational semantics, and hints at the construction which subsequently became known as the canonical model construction. Kaplan claims that it is foreshadowed in Kanger 1957. Presumably he is referring to the section on pp. 36-39, where Kanger defines a relation which holds when everything that is necessary at one place is true at another, but this relation is defined semantically and not through maximal consistent sets, and is not really an anticipation of a Henkin completeness proof. Pp. 13-15 of Hintikka 1957 could also be seen with hindsight as suggesting a Henkin construction.

[4]Kripke 1963b does provide a semantics and an axiomatisation for modal predicate logics, but the paper does not contain any completeness proofs.

which is true in every model and the possible as that which is true in some model', and at the end of that article Bayart is adamant that necessity and validity are quite distinct, though whether his argument there is a good one may be debatable.[5] At the beginning of the 1958 article he explicitly acknowledges that the idea he is trying to formalise is the Leibnizian view of necessity as truth in all possible worlds, and of possibility as truth in some world. He disavows any judgement on the worth of this metaphysics, and simply assumes a set of worlds without saying what they are. Unlike Kripke 1959, Bayart does not make use of a distinguished 'actual' world, though, among the senses of 'valid' that he distinguishes, he does define the validity of a formula in a world in terms of its truth at that world in relation to all interpretations to its variables, which, for Bayart, include individual, propositional and predicate variables. When it comes to proving completeness Bayart is well aware of the limitations of the Henkin method in higher-order logic, and well aware of the sense of validity (which he calls 'quasi-validity') in which a Henkin proof is available.

Bayart lived from 1911-1998. The list of books in the bibliography to Bayart 1958 gives some indication of what he was familiar with. There was a strong tradition of modal logic in Belgium, notably in the work of Robert Feys, and Bayart cites Feys 1950. (The system T in fact comes from Feys 1937, and Feys 1965 shews a tradition familiar with modal logic.) Bayart also lists Lewis's two books, Barcan's articles and Carnap's work. Although the article has a bibliography there are no citations in the text, so, for instance it is not clear whether he had Carnap in mind when he protested against trying to derive necessity from validity. He was also familiar with the λ notation from Church 1940. In Bayart 1959 there is no bibliography, but there is a reference to Henkin in connection with Bayart's completeness proof, which he modestly describes as being 'no more than Henkin's theorem adapted for S5'.

It seems highly likely that Feys's work in modal logic was a significant influence on Bayart. The only work of Feys that Bayart cites is Feys 1950, but in the editorial introduction to Feys 1965 the editor (Joseph Dopp) says on p. vi

> In the course of the years 1948 to 1953 (year wherein occurred the death of McKinsey), Feys repeatedly reworked the part which had fallen to him, «McKinsey acting as advisor». One of these editions was even mimeographed and sent to several different colleagues, who referred to it at times in their writings under the title: FEY'S MCKINSEY, *Modal Logics I.*

Bayart does not refer to this 'edition' in either of the articles included here, but it is difficult to believe that he was not familiar with the material. On p. 152 Feys 1965 there is a rather cryptic reference to a 'lambda function λαM'. Feys's own notation is the circumflex, whereby $\lambda x\alpha$ would be written as $\hat{x}$. In explaining his notation Feys says

> When writing a propositional function under the form of an abstract, we replace each lambda before the variable by a circumflex above the variable.

Feys appears to take the λ-notation as requiring no explanation, and Bayart treats it in the same way when he introduces his letter Z. Bayart's axiomatic basis for S5 is a version of Prior's 1953 basis (see footnote 32) presented in the style of a Gentzen sequent calculus. There is in Feys 1965 an appendix added

[5] Bayart may have been influenced by Kemeny 1956, since Kemeny is critical of Carnap's use of state descriptions (see p.2 of Part I of Kemeny 1956), and suggests a 'new' style of semantics in which expressions are interpreted with reference to models which provide a domain of individuals and functions built up from them. Unfortunately Bayart gives no discussion of the articles listed in his bibliography.

posthumously by Dopp, on pp. 173-185, in which an account is given of various axiomatisations of modal logic in the style of Gentzen 1934. (Professor H. Hubien has informed me that this way of doing logic was very popular in Belgium at the time Bayart was writing.)

There are several reasons for the fact that Bayart's work has not been appreciated as well as it should be by historians of modal logic, at least in the English speaking world. Some of the explanation is that histories of modal logic have tended to ignore modal predicate logic, but there are other factors. Bayart's articles appeared in the first issue of the Belgian journal *Logique et Analyse*, in contrast to the contemporaneous Kripke 1959 which appeared in the US based *Journal of Symbolic Logic*.[6] Second, Bayart does not seem to have persisted with his work on the semantics of modal logic, although he did produce other work in logic, particularly in its application to the philosophy of law. Third, Bayart's notation and terminology, although more common when he wrote, are less common today, and his formal work makes for difficult reading. Fourth, and perhaps most important in the development of modal logic in the international community of logicians, it was published in French. The present article contains an English translation, using more common notation, of Bayart's two articles.

2. Remarks on the translation[7]

There is always a tension in translating a work like this. On the one hand there is the demand to be faithful to the original, and on the other hand there is the demand to make the translation as accessible to the audience as possible. This is made more difficult in Bayart's case by two features. The first is that the work was published in 1958 and 1959, in the very earliest days of the development of the semantic study of modal logic. Indeed that is the principal reason for its importance, as I have explained in the historical introduction. Because of this, some of Bayart's terminology may seem strange to modern readers. Thus for instance, Bayart uses the term 'proposition', where we would now use the word 'formula', for the linguistic item which expresses a proposition in a logical language. But he uses the term 'predicate', not for a linguistic item, but for its value — particularly in the expression 'n-place intensional predicate'. I have adopted the word 'formula' — sometimes 'well-formed formula or wff — for Bayart's 'proposition', and I have used the phrase 'n-place intensional relation' for Bayart's 'n-place predicate'. I have however retained Bayart's 'propositional variable' and 'predicate variable', since these are still in common use. The second feature which has caused problems in making a translation accessible is that the original French text seems to have been set from a ms which had no logic symbols or italicising or subscripting. It is not clear how much Bayart's choice of notation was determined by his typewriting facilities. For instance his use of the Polish notation may be as much for historical reasons — it is used for instance in Feys 1950, where you find what seems to be the first published use of L as a necessity operator — as for typographical reasons. I have adapted Bayart's notation, first by adopting a modified Russellian, rather than a Polish notation, except for retaining Bayart's L and M for necessity and possibility, and second by making extensive use of Greek letters and of italicisation, subscripting and superscripting. For those who wish to consult Bayart's own terminology and notation we have put onto a website both the original French version and a version of this translation in which Bayart's own notation and terminology are retained.[8]

Here are some of the specific changes found in the present version. To a considerable extent I have expressed Bayart's formal passages in the notation of Hughes and Cresswell 1996. For Bayart's N, K, A,

[6]Although Kripke 1959 lists no institutional affiliation Kripke was subsequently at Harvard and at Princeton.

[7]In this section, and whet follows the personal pronouns 'I', 'me' and so on, refer to Cresswell.

[8]This may be found at www.***

C and *E*, I have used ~, ∧, ∨, ⊃ and ≡. For Bayart's *Px* and *Sx* I have used ∀*x* and ∃*x*. (Bayart takes all these as primitive, which leads to some repetitions in his proofs, especially those by induction on the construction of wff, but I have not changed this.) For Bayart's abstraction symbol *Z* I have used the standard λ. (Bayart himself points out that he is using *Z* in place of 'lambda'.) As metavariables for wff (well-formed formulae) I have used α, β etc. For predicate variables I have followed Whitehead and Russell 1910 and used φ, ψ, χ, etc. I have followed Bayart in using 'universe' rather than Kripke's 'model structure' or Scott's 'frame', but have referred to a domain D of individuals and a set W of possible worlds. I have referred to the members of W as *w*, *w*′, etc, rather than as M, M′ etc. In place of Bayart's 'value system S' I have spoken of an interpretation V, and where Bayart would say that 'a proposition *f* is true for value system UMS' I have frequently written V(α,*w*) = T, it being understood that V is relative to D and W. (Where both ⟨D,W⟩ and V are clear, I have sometimes written simply 'α is true in *w*'.) Otherwise the terminology in this translation is explained explicitly in footnotes or made clear by the context. In the 1958 article Bayart uses many short paragraphs separated by a line space, and does not indent the fist word. In the 1959 article new paragraphs within a section begin on the next line with the first word indented. I have followed Bayart's setting out for easy reference, unless clarity demands otherwise, so that a comparison between the translation here and the version on the website or the original French should not be difficult. Part of this translation was begun in the early 1960s when I was making a survey of work then available in modal logic, in preparation for what became Hughes and Cresswell 1968.[9] This translation lay dormant until I had the opportunity of a residential Fellowship with the Flemish Institute for Advanced Studies of the Royal Flemish Academy of Belgium for Science and the Arts in the latter part of 2010.

3. Bayart 1958[10]

THE SOUNDNESS OF FIRST AND SECOND-ORDER S5 MODAL LOGIC

I Semantic definitions

0. To formulate a semantic theory of modal logic it is not sufficient to define for example, the necessary as that which is true in every model and the possible as that which is true in some model. These definitions would do no more than introduce the notions of 'necessary' and 'possible' in the metalanguage. A semantics of modal logic demands that we assume an object language containing modal symbols and that we define under what conditions to attribute the values 'true' or 'false' to the formulae of this object language.

One can then very easily define the validity and satisfiability of formulae in this language and shew the soundness of such and such a deductive system, this soundness consisting in the fact that all derivable formulae in the considered systems are valid.

It is a theory of this kind which we propose to develop in the present article, inspired by the Leibnizian

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[9]A generalisation of Bayart's completeness proof to the system T appeared in Cresswell 1967 and later in Hughes and Cresswell 1968. A more recent proof method for systems with the Barcan Formula is found in Thomason 1970.

[10]Translation by M.J. Cresswell of 'La correction de la logique modale du premier et second ordre S5', *Logique et Analyse*, 1, 1958, pp. 28–44. In this version I have corrected obvious typos. Some of these are indicated in the website version in square brackets [..]. I have changed Bayart's notation in this version as explained in the introduction or commentary or in footnotes. (All footnotes are my comments on the translation.)

definition of necessity as truth in all possible worlds.

It is not, in our opinion, the task of the logician to examine the value of this Leibnizian metaphysics. We can confine ourselves to shewing that if one takes this metaphysics one can formulate for the modal logic S5 a semantical theory analogous to the formal semantic theories of non-modal logic.

The modal semantic theory leads us to consider relations of a special kind. These are relations whose extension varies in one world or another, and we give them the name of 'intensional relations'.

1. Letting D and W be two non-empty sets, not having any common elements, call D the 'set of individuals' and W the 'set of worlds'. We say that these sets D and W constitute a universe ⟨D,W⟩, For each natural number n we mean by 'n-place intensional relation' a function of n+1 arguments, taking the value T, 'true', or F, 'false', having a world as its first argument and for n ≠ 0, having n individuals as its last n arguments.

Letting *a* and *b* be the cardinal numbers of D and W, for any natural number n there are $c = 2^{ba^{\mathrm{n}}}$ n-place intensional relations.

2. We assume a language $\mathfrak{L}$. For the moment we confine ourselves to considering a language without axioms or rules of deduction. This language contains a denumerable infinity of individual variables, and for each natural number n, a denumerable infinity of n-place predicate variables. It does not contain constants for individuals or predicates.

In the following exposition the different types of variables will be designated by small letters which play the rõle of syntactical variables. Certain Greek letters can also designate other expressions than variables. We indicate each time in the context what sort of expressions are designated by the syntactical variables. These syntactical variables may be followed by numbers, and we write e.g., $x_0, x_1, x_2, \ldots, x_{\mathrm{n}}$.

We shall adopt the following notation. The language $\mathfrak{L}$ contains the symbols ~, ∧, ∨, ⊃, and ≡ for negation, conjunction, disjunction, implication and equivalence, the symbols ∀ and ∃ for the universal and existential quantifiers, and the symbols *L* and *M* for necessity and possibility. We introduce these symbols not only in the object language but also in the metalanguage, where they are combined with syntactical variables to form complex syntactical expressions.

Formation rules are as follows:

a.) a 0-place predicate variable is a wff (well-formed formula.)
b.) an n-place predicate variable followed by n individual variables is a wff.
c.) If α is a wff ~α is a wff.
d.) If α and β are wff then (α ∧ β) is a wff
e.) If α and β are wff then (α ∨ β) is a wff
f.) If α and β are wff then (α ⊃ β) is a wff
g.) If α and β are wff then (α ≡ β) is a wff

h.) If α is a wff and x is a variable then $\forall x\alpha$ is a wff[11]
i.) If α is a wff and x is a variable then $\exists x\alpha$ is a wff
j.) If α is a wff $L\alpha$ is a wff.
k.) If α is a wff $M\alpha$ is a wff.
l.) There are no other wff

We have thus a pure modal second-order language.

3. Let $\langle D,W\rangle$ be a universe composed of the set D of individuals and W of worlds. We agree that the variables for individuals of the language $\mathcal{L}$ can take as values individuals of the set D and that for each natural number n the variables for n-place predicates take as values n-place intensional relations defined on the universe $\langle D,W\rangle$.

We take a universe $\langle D,W\rangle$, a world w of this universe and an interpretation V relative to this universe. We then define the notions 'true for universe $\langle D,W\rangle$, the world w and the interpretation V', and 'false for universe $\langle D,W\rangle$, the world w and the interpretation V'.[12] Let α be a wff of language $\mathcal{L}$.

If α is a variable p for 0-place predicates, then if ω is the 0-place intensional relation given by V as the value of p, $V(\alpha,w) = \omega(w)$.

If α is $\varphi x_1...x_n$, where φ is an n-place predicate variable ($n \neq 0$) and where $x_1,...,x_n$ are individual variables if ω, $a_1,..., a_n$ are respectively the n-place intensional relation and the individuals given as values of φ, $x_1,..., x_n$, $V(\alpha,w) = \omega(w,a_1, ..., a_n)$.

If α has the form $\sim\beta$, where β is a wff, $V(\alpha,w) = T$ if $V(\beta,w) = F$, and $V(\alpha,w) = F$ if $V(\beta,w) = T$.

If α has the form $\beta \wedge \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta w) = V(\gamma,w) = T$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\beta \vee \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta,w) = T$ or if $V(\gamma,w) = T$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\beta \supset \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta,w) = F$ or if $V(\gamma,w) = T$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\beta \equiv \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta w) = V(\gamma,w)$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\forall x\beta$, where β is a wff and x a variable for individuals or predicates, $V(\alpha,w) = T$ if, for each interpretation V' relative to $\langle D,W\rangle$ which gives to all the variables other than x the same values as

[11] Where, as here, variables of all kinds are intended, individual variables, propositional variables and predicate variables, I have followed Bayart in using a single letter (I have used x where Bayart uses v.) But where it is clear that a propositional variable is intended I have used p, and where it is understood that a predicate variable is intended I have used φ.

[12] I am using $V(\alpha,w) = T$ as an abbreviation for Bayart's 'α is true according to w and V', and $V(\alpha,w) = F$ as an abbreviation for Bayart's 'α is false according to w and V'. Bayart spells it out each time.

V, $V'(\beta,w) = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $\exists x\beta$, where β is a wff and *x* a variable for individuals or predicates, $V(\alpha,w) = T$ if there is an interpretation V′ relative to $\langle D,W\rangle$ which gives to all the variables other than *x* the same values as V, and $V'(\beta,w) = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $L\beta$, where β is a wff, $V(\alpha,w) = T$ if for every world *w′* of the universe $\langle D,W\rangle$, $V(\beta,w') = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $M\beta$, where β is a wff, $V(\alpha,w) = T$ if there is a world *w′* of the universe $\langle D,W\rangle$ such that $V(\beta,w') = T$, and otherwise $V(\alpha,w) = F$.

4. We take a universe $\langle D,W\rangle$ and a world *w* of this universe. We define for formulae of the language $\mathfrak{L}$ the notions of 'valid in $\langle\langle D,W\rangle,w\rangle$' and 'satisfiable in $\langle\langle D,W\rangle,w\rangle$'. Let α be a wff of $\mathfrak{L}$.

The wff α will be valid in $\langle\langle D,W\rangle,w\rangle$ if and only if, for each interpretation V relative to $\langle D,W\rangle$, $V(\alpha,w) = T$.

The wff α will be satisfiable in $\langle\langle D,W\rangle,w\rangle$ iff there is an interpretation V relative to $\langle D,W\rangle$ such that $V(\alpha,w) = T$.

The wff α will be valid in $\langle D,W\rangle$ iff it is valid in every $\langle\langle D,W\rangle,w\rangle$ (for every world *w*).

The wff α will be satisfiable in $\langle D,W\rangle$ iff there is some world *w* such that α is satisfiable in $\langle\langle D,W\rangle,w\rangle$.

We define for the language $\mathfrak{L}$ the notions 'valid' and 'satisfiable'.

The wff α will be valid iff it is valid in all universes.

The wff α will be satisfiable iff it is satisfiable in some universe.

We transform the language $\mathfrak{L}$ into a system of deduction D_{S5} by giving axioms and rules of deduction. D_{S5} will be sound if one can only prove valid formulae. D_{S5} will be complete if one can prove any valid formula of the language $\mathfrak{L}$.

II *Auxiliary language*

5. From the expressions of the language $\mathfrak{L}$ we form an auxiliary language $\mathfrak{L}'$ by introducing the symbol λ.

The expressions of $\mathfrak{L}'$ will play a syntactical role and so appear in the metalanguage. They designate certain expressions of $\mathfrak{L}$ which will be called the resultants of corresponding expressions of $\mathfrak{L}'$.

In the exposition which follows we continue to use small letters to indicate syntactical variables and combine them with the logical constants ~, ∧, ∨, ⊃, ≡, ∀, ∃, *L*, *M* and the operator λ to form complex syntactical expressions.

The symbol λ followed by a finite number n ($n \neq 0$) of individual variables is an n-place individual abstractor.

The symbol λ followed by a 0-place predicate variable is a propositional abstractor.

The symbol λ followed by an n-place predicate variable ($n\neq0$) is an n-place predicate abstractor.

An n-place individual abstractor followed by a wff is an n-place individual abstract.[13]

A propositional abstractor followed by a wff is a propositional abstract.

A predicate abstractor followed by a wff is a predicate abstract.

An n-place individual abstract followed by n individual variables (not necessarily distinct) is a primary, simple, paraformula.[14]

The expression obtained by substituting, in any way whatever in a wff, n-adic individual abstracts for n-adic predicate variables is a primary complex paraformula.

One sees that in a complex primary paraformula the individual abstracts of n_0, n_1, n_2, ... variables for individuals are followed respectively by the n_0, n_1, n_2, ... variables for individuals which follow the variables for predicates of n_0, n_1, n_2, ... places in the original wff. These individual abstracts will thus form, with variables for individuals, simple primary paraformulae.

A propositional abstract followed by a wff is a propositional secondary paraformula. A predicate abstract where the abstractor is an n-place predicate variable ($n\neq0$) followed by an n-place individual abstract is a predicate secondary paraformula.

(Note: In the exposition which follows we introduce parentheses into expressions of the auxiliary language for ease of reading.)

6. In an abstract the free variables are the variables which occur free in the formula which follows the abstractor, other than the variables in the abstractor.

In an abstract the bound variables are the variables of the abstractor, the bound variables which follow the abstractor and the free variables of this formula which occur also in the abstractor. One says of these last variables that they are bound by the abstractor. In particular in an individual abstract $\lambda x_1...x_n(\alpha)$, where α is a wff, if the variable x_i ($i = 1,2,...n$) appears free in α it is said to be bound by the i-th variable of the

[13]Bayart calls an n-place individual abstract a 'parapredicate'. He adds 'or an n-place individual abstract.'

[14]I have used 'paraformula' for Bayart's 'paraproposition' in line with my use of 'formula' or 'wff' for Bayart's 'proposition'. A paraformula is an expression of $\mathscr{L}'$ which is not a wff of $\mathscr{L}$, though its resultant is. I explain what is going on here in more detail in the commentary.

abstractor.

The resultant of a simple primary paraformula $\lambda x_1 ... x_n(\alpha)y_1 ... y_n$ is the wff α' which is obtained by simultaneously substituting in the wff α individual variables $y_1,...,y_n$ for the individual variables $x_1,...,x_n$ wherever they occur free in α.

Substitution is simultaneous if at each place in α where a variable occurs bound by the abstractor one makes one and only one substitution.[15]

The resultant of a complex primary paraformula is a wff α' which one obtains by replacing in α each simple primary paraformula by its resultant.

The resultant of a propositional paraformula $\lambda p(\alpha)\beta$ is the wff α'_j which is obtained by substituting in the wff α the wff β for the propositional variable p wherever it occurs free in α.

The intermediate resultant of a predicate paraformula $\lambda\varphi(\alpha)\theta$, where θ is an individual abstract of the same number of places as the variable φ, is the complex primary paraformula obtained by substituting in the wff α the individual abstract θ for the variable φ wherever the latter occurs free in α.[16]

The final resultant, or more briefly the resultant, of a predicate paraformula is the resultant of the intermediate resultant.

7. A simple primary paraformula $\lambda x_1 ... x_n(\alpha)y_1 ... y_n$ is well-formed if for every i (i = 1, 2, ...n) the variable x_i does not occur free in α in the scope of a quantifier $\forall y_i$ or $\exists y_i$. A complex primary paraformula is well-formed if all its simple primary paraformulae are well-formed.

A propositional paraformula $\lambda p(\alpha)\beta$ is well-formed if the variable p does not occur free in α within the scope of a quantifier $\forall x$ or $\exists x$ where x is any variable which occurs free in β.

A predicate paraformula $\lambda\varphi(\alpha)\theta$ is well-formed if

1. The variable φ does not occur free in α in the scope of a quantifier $\forall x$ or $\exists x$ where x is any variable which occurs free in the individual abstract θ

and if also

2. The intermediate resultant of $\lambda\varphi(\alpha)\theta$ is well-formed.

III Sematic properties of paraformulae

[15] I take it that by 'simultaneous' Bayart means 'uniform' in the sense that the same variable must be replaced on each occurrence by the same expression.

[16] I have used θ as a metavariable for an individual abstract, or on occasions for a predicate or propositional abstract.

8. Definition. We give a recursive definition of the notion 'modalised wff'.[17]

1. Formulae of the form $L\alpha$ and $M\alpha$ are modalised
2. If α is a modalised wff then $\sim\alpha$ is a modalised wff.
3. If α and β are modalised wff then $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \supset \beta$ and $\alpha \equiv \beta$ are modalised wff.
4. If α is a modalised wff and if x is a variable then $\forall x\alpha$ and $\exists x\alpha$ are modalised wff.
5. There are no other modalised wff than those defined by rules 1-4.

9. Definition. The value of an individual abstract $\lambda x_1 ... x_n(\alpha)$ for a universe $\langle D,W \rangle$ and an interpretation V relative to $\langle D,W \rangle$ is the n-place intensional relation which, for every world w and any series of individuals $a_1, ..., a_n$, takes the value T or F according as $V'(\alpha,w) = T$ or F, where V′ is an interpretation which assigns individuals $a_1,..., a_n$ as values to the individual variables $x_1,..., x_n$ respectively and which gives to all other variables the same values as V.

10. *Theorem I*: Consider a universe $\langle D,W \rangle$, two worlds w and w' of W and any interpretation V relative to $\langle D,W \rangle$. If α is a modalised wff then $V(\alpha,w) = V(\alpha,w')$.

Proof by induction on the definition of modalised wff.

11. *Theorem II*: Let α be a wff containing only $x_1,..., x_n$ as free variables. Consider any universe $\langle D,W \rangle$, a world w of W and two interpretations V and V′ relative to $\langle D,W \rangle$ which do not differ in the values assigned to $x_1,..., x_n$. Then $V(\alpha,w) = V'(\alpha,w)$. In particular if α is a closed wff (i.e., does not contain free variables) then for any two interpretations V and V′ relative to $\langle D,W \rangle$, $V(\alpha)(w) = V'(\alpha)(w)$.

Proof by induction on the construction of α.

12. *Theorem III*: Let θ be an individual abstract $\lambda x_1 ... x_r(\alpha)$ which contains only the variables $y_1,..., y_n$ free. Take any universe $\langle D,W \rangle$, any world w of W and any two interpretations V and V′ relative to $\langle D,W \rangle$ which do not differ in the values given to the variables $y_1,..., y_n$. Then $V(\theta) = V'(\theta)$. In particular if θ is a closed abstract (i.e., does not contain free variables) then for any two interpretations V and V′ relative to $\langle D,W \rangle$, $V(\theta) = V'(\theta)$.

In the proof we rely on the definition of an individual abstract and on the result of theorem II.

13. *Theorem IV*: For any universe $\langle D,W \rangle$, and world w of W and any interpretation V relative to $\langle D,W \rangle$, if $\lambda x_1 ... x_n(\alpha) y_1 ... y_n$ is a well-formed simple primary paraformula and α' is the resultant of this formula then $V(\alpha',w) = V'(\alpha,w)$, where V′ is an interpretation which gives to the individual variables $x_1,...,x_n$ the individuals $a_1,..., a_n$ respectively, being the same individuals as assigned by V to the variables $y_1,..., y_n$ respectively, and which gives all other variables the same values as V does.

Proof by induction on the construction of α.

[17] I have translated Bayart's «proposition couverte» here not as 'closed formula' but as 'modalised formula', since this is in accordance with standard usage in modal logic. Strictly speaking it should probably be 'completely modalised formula'

14. *Theorem V*: For any universe $\langle D,W\rangle$, any world w of W, and any interpretation V relative to $\langle D,W\rangle$, if $\lambda x_1...x_n(\alpha)y_1...y_n$ is a well-formed simple primary paraformula, and if ω is the n-place intensional relation which is the value given by V to the individual abstract $\lambda x_1...x_n(\alpha)$, the value $V(\alpha',w)$ of the resultant α' of this paraformula will be $\omega(w,a_1,..., a_n)$ where $a_1,..., a_n$ are the values given by V to the variables $y_1,..., y_n$.

The proof relies on the definition of an individual abstract and the result of theorem IV.

15. *Theorem VI*: For any universe $\langle D,W\rangle$, any world w of W, and any interpretation V relative to $\langle D,W\rangle$, if $\lambda p(\alpha)\beta$ is a well-formed propositional paraformula and α' is its resultant then $V(\alpha',w) = V'(\alpha,w)$, where V′ is the interpretation such that $V'(p)(w) = (\beta,w)$, and which gives all the other variables the same value as V.

Proof by induction on the construction of α.

16. *Theorem VII*: For any universe $\langle D,W\rangle$, any world w of W, and any interpretation V relative to $\langle D,W\rangle$, if $\lambda\varphi(\alpha)\theta$ is a well-formed predicate paraformula where φ is an n-place predicate variable and θ is an n-place individual abstract, and α' is the final resultant of this paraformula, then $V(\alpha',w) = V'(\alpha,w)$, where V′ is the interpretation which assigns to the variable φ the value which the individual abstract θ takes for V, and which gives all the other variables the same values as V.

Proof by induction on the construction of α.

IV Soundness of the second-order S5.

17. We formulate S5 by means of Gentzen systems. A sequent comprises: first a finite series,[18] possibly empty, of formulae of the language $\mathfrak{L}$, which is called the 'antecedent'; second the symbol $\vdash$, and third a finite series, possibly empty, of formulae of the language $\mathfrak{L}$, called the 'consequent'.[19]

The system S5 comprises an axiom schema and twenty eight rules of deduction divided into four groups: structural rules, propositional rules, quantificational rules and modal rules. The rules permit the passage from one or two sequents called premises to another sequent called the conclusion.

To formulate the axiom schema and the rules of deduction we use a metalanguage containing, among other things, the expressions which we used in formulating the theory of paraformulae. We will also include the symbol $\vdash$ in the metalanguage.

In particular, in the present section IV, the letters α and β will designate formulae and the letters Φ, Φ' and Δ, Δ' will designate series of formulae. In an expression of the form $\forall x\alpha$ or $\exists x\alpha$, the letter x

[18] I have retained Bayart's word «series» here, though perhaps 'sequence' would be more appropriate, and I have used 'sequence' in the commentary. I have translated Bayart's word «sequence» in the context of Gentzen systems as 'sequent'.

[19] I have translated Bayart's «conséquent» as 'consequent', although the term 'succedent' is sometimes used for Gentzen's 'Sukzedens'. 'Succedent' is used in Kanger 1957, and in Szabo's English translation of Gentzen's papers. Feys and Ladrière 1955 also uses 'conséquent'. I have used $\vdash$ rather than Gentzen's → because the letter is easily confused with a propositional operator. Both 'consequent' and $\vdash$ are used in Dopp's appendix to Feys 1965,

designates an individual variable, a propositional variable or a variable for an n-place predicate. ($n > 0$) In an expression of the form $\lambda x(\alpha)\theta$ the letter θ designates an individual variable, a wff or an n-place individual abstract, according as x is an individual variable, a propositional variable or an n-place predicate variable. An expression of the form $\lambda x(\alpha)\theta$ designates a paraformula, but it is understood that it is not the paraformulae but the resultants of the paraformulae which figure in deductions.

In the antecedent and consequent, expressions designating formulae or series of formulae are separated by commas.

18. We define the notions 'true' and 'false' for Gentzen sequents relative to a universe $\langle D,W\rangle$, a world w and an interpretation V.

A sequent $\Phi \vdash \Delta$ is true in w if Φ contains a wff false in w or if Δ contains a wff true in w. Otherwise the sequent $\Phi \vdash \Delta$ is false in w.

Following from this definition we can, as we have done in paragraph 4 for wff, define for sequents the notions 'valid for $\langle\langle D,W\rangle,w\rangle$', 'valid for $\langle D,W\rangle$', 'valid', 'satisfiable for $\langle\langle D,W\rangle,w\rangle$', 'satisfiable for $\langle D,W\rangle$', 'satisfiable'.

We shew that the system S5 is sound in the sense that all deductions are valid sequents. We shew, in particular, that the axioms are valid, and that the rules of deduction are such that, if the premises are valid, the conclusion is valid. It is convenient here to recall that 'valid' is synonymous with 'true for every universe $\langle D,W\rangle$, every world w of this universe, and every interpretation V relative to this universe'.

We shew the soundness of the axiom (or, what comes to the same thing, the axiom schema) and the rules of deduction as we present them.

19. The axiomatic schema (which we label 'Ax') is the following

$$\alpha \vdash \alpha$$

The axioms which are instances of this schema are obviously valid. If α designates a true formula, Δ contains a true formula and if α designates a false formula then Φ contains a false formula.[20]

There are seven structural rules; to be precise, addition, permutation and contraction in the antecedent (designated, respectively, by 'ADI, 'PEI', and 'COI'), addition, permutation and contraction in the consequent (designated, respectively, by 'IAD, 'IPE', and 'ICO') and cut (designated by 'Cut').

The rules are as follows:

[20]In this case Φ and Δ are both simply α.

$$\text{ADI}\ \frac{\Phi \vdash \Delta}{\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta}{\Phi \vdash \Delta,\alpha}\ \text{IAD}$$

$$\text{PEI}\ \frac{\Phi,\alpha,\beta,\Phi' \vdash \Delta}{\Phi,\beta,\alpha,\Phi' \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta,\alpha,\beta,\Delta'}{\Phi \vdash \Delta,\beta,\alpha,\Delta'}\ \text{IPE}$$

$$\text{COI}\ \frac{\alpha,\alpha,\Phi \vdash \Delta}{\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta,\alpha,\alpha}{\Phi \vdash \Delta,\alpha}\ \text{ICO}$$

$$\frac{\Phi \vdash \Delta,\alpha \qquad \alpha,\Phi \vdash \Delta}{\Phi \vdash \Delta}\ \text{Cut}$$

The soundness of the rules with one premise is obvious. The proof of the soundness of Cut is as follows. Assume a universe ⟨D,W⟩, a world *w* in this universe, and an interpretation V in this universe. By hypothesis the two premises are true in *w*. So Φ will contain a formula false in *w* or Δ will contain a formula true in *w*, for otherwise α would have to be true for the first premise to be true, and α would have to be false for the second premise to be true. It follows that the conclusion is true in *w*.

20. There are ten propositional rules; to be precise, the introduction of ~, ∧, ∨, ⊃ and ≡ in the antecedent (designated respectively by '~I', '∧I', '⊃I' and '≡I', and the introduction of ~, ∧, ∨, ⊃ and ≡ in the consequent (designated respectively by 'I~', 'I∧', 'I⊃' and 'I≡'.

The rules are as follows:

$$\text{\textasciitilde I}\ \frac{\Phi \vdash \Delta,\alpha}{\sim\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\alpha,\Phi \vdash \Delta}{\Phi \vdash \Delta,\sim\alpha}\ \text{I\textasciitilde}$$

$$\wedge\mathrm{I}\ \frac{\alpha,\beta,\Phi \vdash \Delta}{\alpha \wedge \beta,\Phi \vdash \Delta} \qquad \frac{\Phi \vdash \Delta,\alpha \qquad \Phi \vdash \Delta,\beta}{\Phi \vdash \Delta,\alpha \wedge \beta}\ \mathrm{I}\wedge$$

$$\vee\mathrm{I}\ \frac{\alpha,\Phi \vdash \Delta \qquad \beta,\Phi \vdash \Delta}{\alpha \vee \beta,\Phi \vdash \Delta} \qquad \frac{\Phi \vdash \Delta,\alpha,\beta}{\Phi \vdash \Delta,\alpha \vee \beta}\ \mathrm{I}\vee$$

$$\supset\mathrm{I}\ \frac{\Phi \vdash \Delta,\alpha \qquad \beta,\Phi \vdash \Delta}{\alpha \supset \beta,\Phi \vdash \Delta} \qquad \frac{\alpha,\Phi \vdash \Delta,\beta}{\Phi \vdash \Delta,\alpha \supset \beta}\ \mathrm{I}\supset$$

$$\equiv\mathrm{I}\ \frac{\Phi \vdash \Delta,\alpha,\beta \qquad \alpha,\beta,\Phi \vdash \Delta}{\alpha \equiv \beta,\Phi \vdash \Delta} \qquad \frac{\alpha,\Phi \vdash \Delta,\beta \qquad \beta,\Phi \vdash \Delta,\alpha}{\Phi \vdash \Delta,\alpha \equiv \beta}\ \mathrm{I}\equiv$$

We prove the soundness of ≡I and I≡

For ≡I: Consider a universe ⟨D,W⟩, a world *w* and an interpretation V relative to this universe. If Φ contains a formula false in *w*, or if Δ contains a formula true in *w* the conclusion is true in *w*. If Φ does not contain a formula false in *w* and if Δ does not contain a formula true in *w*, then, since by hypothesis the first premise is true in *w*, it is necessary that one of the formulae α or β is true in *w*, and, since by hypothesis the second premise is true in *w* it is necessary that one of the formulae α or β is false in *w*. If one of the two formulae α and β is true and the other is false, α ≡ β will be false in *w*, and so the conclusion will be true in *w*.

For I≡: Consider a universe ⟨D,W⟩, a world *w* and an interpretation V relative to this universe. If Φ contains a formula false in *w*, or if Δ contains a formula true in *w* the conclusion is true in *w*. If Φ does not contain a formula false in *w* or if Δ does not contain a formula true in *w*, two cases are possible: If α is true in *w* then, since by hypothesis the first premise is true in *w* it is necessary that β will be true in *w*. If α is false in *w* then, since by hypothesis the second premise is true in *w* it is necessary that β is false in *w*; if α and β are true in *w* or if α and β are false in *w*. α ≡ β is true in *w*, and so the conclusion will be true in *w*.

21 There are four rules of quantification; to be precise the introduction of ∀ and ∃ in the antecedent (designated, respectively, by '∀I', '∃I') and the introduction of ∀ and ∃ in the consequent (designated, respectively, by 'I∀', 'I∃').

The rules are as follows:

$$\forall\mathrm{I}\ \frac{\lambda x(\alpha)\theta,\Phi \vdash \Delta\text{[21]}}{\forall x\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta,\alpha}{\Phi \vdash \Delta,\forall x\alpha}\ \mathrm{I}\forall$$

$$\exists\mathrm{I}\ \frac{\alpha,\Phi \vdash \Delta}{\exists x\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta,\lambda x(\alpha)\theta}{\Phi \vdash \Delta,\exists x\alpha}\ \mathrm{I}\exists$$

Restriction (1): In the rules ∀I and I∃, $\lambda x(\alpha)\theta$ must be a well-formed paraformula.

Restriction (2): In the rules I∀ and ∃I the variable x cannot occur free in the formulae of Φ or of Δ.

We prove the soundness of ∀I and of I∀.

For ∀I: Consider a universe ⟨D,W⟩, a world w and an interpretation V relative to this universe. If Φ contains a formula false in w, or if Δ contains a formula true in w the conclusion is true in w. If Φ does not contain a formula false in w and if Δ does not contain a formula true in w, then, since by hypothesis the first premise is true in w, $\lambda x(\alpha)\theta$ will be false in w. It follows by virtue of theorems IV, VI or VII that there is an interpretation V′, which gives the value to x that V gives to θ, and which gives all other variables the same value that V gives, and is such that α is false in w'. There is thus an interpretation V′, which is no different from V except for the value given to x, and is such that α is false in w'. So $\forall x\alpha$ is false in w, and so the conclusion will be true in w.

For I∀: Consider a universe ⟨D,W⟩, a world w and an interpretation V relative to this universe. If Φ contains a formula false in w, or if Δ contains a formula true in w the conclusion is true in w. If Φ does not contain a formula false in w and if Δ does not contain a formula true in w, since the formulae in Φ and in Δ do not contain free x, it follows, in virtue of theorem II that, for every interpretation V′ which gives to all variables except x the same values as V, Φ will not contain formulae which are false in w with respect to V′, and that Δ will not contain formulae which are true in w with respect to V′. So, by hypothesis, for all these interpretations V′, the premises are true in w'. Thus, for all these interpretations $V'(\alpha,w) = T$. Thus $V(\forall x\alpha,w) = T$, and so the conclusion will be true in w.

22. There are four modal rules; to be precise the introduction of L and M in the antecedent (designated, respectively, by ‘LI, ‘MI’) and the introduction of L and M in the consequent (designated, respectively, by ‘IL’, ‘IM’).

[21]In ∀I x indicates a variable of any type, individual, propositional or predicate, and θ indicates that the argument of $\lambda x(\alpha)$ can be a complex abstract of $\mathfrak{L}'$ of the same type as x. Of course (the resultant of) θ can also be another variable, as it always is when x is an individual variable. (Recall that Bayart is explicit that it is the resultants of the $\mathfrak{L}'$ expressions which are the wff of $\mathfrak{L}$ which are involved in these rules.)

The rules are as follows:

$$\text{LI}\ \frac{\Phi \vdash \Delta,\alpha}{L\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\alpha,\Phi \vdash \Delta}{\Phi \vdash \Delta,L\alpha}\ \text{IL}$$

$$\text{MI}\ \frac{\alpha,\Phi \vdash \Delta}{M\alpha,\Phi \vdash \Delta} \qquad\qquad \frac{\Phi \vdash \Delta,\alpha}{\Phi \vdash \Delta,M\alpha}\ \text{IM}$$

Restriction (3): In the rules IL and MI the formulae of Φ and Δ must be fully modalised.

We prove the soundness of LI and of IL

For LI: Consider a universe ⟨D,W⟩, a world *w* and an interpretation V relative to this universe. If Φ contains a formula false in *w*, or if Δ contains a formula true in *w* the conclusion is true in *w*. If Φ does not contain a formula false in *w* and if Δ does not contain a formula true in *w*, then, since by hypothesis the first premise is true in *w*, it is necessary that α will be false in *w*. So *L*α is false in *w* and the conclusion will be true in *w*.

For IL: Consider a universe ⟨D,W⟩, a world *w* and an interpretation V relative to this universe. If Φ contains a formula false in *w*, or if Δ contains a formula true in *w* the conclusion is true in *w*. If Φ does not contain a formula false in *w* and if Δ does not contain a formula true in *w*, since the formulae in Φ and in Δ are fully modalised, it follows, in virtue of theorem I that, for every world *w*′, Φ will not contain formulae which are false in *w*′, and that Δ will not contains formulae which are true in w′. Now, by hypothesis, for all worlds w′, the premises are true in *w*′. Thus, for all worlds w′, α will be true in *w*′. Thus *L*α is true in *w*, and so the conclusion will be true in *w*.

V First-order logic

23. From the preceding one can easily extract the theory of first-order modal logic.

First-order modal logic contains a denumerable infinity of individual variables, and, for each natural number n, a denumerable infinity of variables for n-place predicates.

The formation rules are the same as for second-order logic, except that, in expressions of the form ∀*x*α or ∃*x*α, *x* must be an individual variable.

24. The semantic definitions are the same as for second-order modal logic.

25. In first-order modal logic we need only consider abstractors containing just an individual variable, and so simple primary paraformulae of the form λ*x*(α)*y*, where *x* is an individual variable.

26. We only need theorems I, II and IV. In the last theorem we only need to consider paraformulae formed by abstractors containing a single individual variable.

27. The deduction rules are the same as in second-order logic but the scope of the quantification rules is automatically reduced, when we note that, in expressions of the form $\forall x\alpha$, $\exists x\alpha$ and $\lambda x(\alpha)y$, x designates an individual variable, and excludes predicate variables.

The soundness of first-order modal logic can be proved in the same way as in second-order modal logic.

VI Necessity and validity

28. One might perhaps combine the notions of necessity and validity. One might then formulate the following semantic theory:

Instead of providing a universe consisting of a domain and a set of individuals one might simply give a domain D, i.e., a set of individuals. One then gives a set of extensional relations. For each natural number n, an extensional relation is a function of n arguments, these arguments being individuals, and able to take the values T or F.

Individual variables can take individuals as values, and n-place predicate variables can take n-place extensional relations as values. Propositional variables can take T or F as values.

29. Assume a domain D and an interpretation V.

A propositional variable p is true or false in $\langle D,V\rangle$ according as $V(p) = T$ or F.

A wff of the form $\varphi x_1 ... x_n$, where φ is an n-place predicate variable and $x_1,...,x_n$ are individual variables, will be true in $\langle D,V\rangle$ if, where $V(\varphi)$ is the extensional relation ω and the individuals $a_1,..., a_n$ are the values given in this order to $x_1,..., x_n$ respectively, $\omega(a_1,..., a_n) = T$; and $\varphi x_1 ... x_n$ will be false in $\langle D,V\rangle$ if $\omega(a_1,..., a_n) = F$.

A wff of the form $\sim\alpha$ is true in $\langle D,V\rangle$ if α is false in $\langle D,V\rangle$, and otherwise false in $\langle D,V\rangle$.

A wff of the form $\alpha \wedge \beta$ is true in $\langle D,V\rangle$ if α and β are true in $\langle D,V\rangle$, and otherwise false in $\langle D,V\rangle$.

A wff of the form $\alpha \vee \beta$ is true in $\langle D,V\rangle$ if α is true in $\langle D,V\rangle$ or β is true in $\langle D,V\rangle$, and otherwise false in $\langle D,V\rangle$.

A wff of the form $\alpha \supset \beta$ is true in $\langle D,V\rangle$ if α is false in $\langle D,V\rangle$ or β is true in $\langle D,V\rangle$, and otherwise false in $\langle D,V\rangle$.

A wff of the form $\alpha \equiv \beta$ is true in $\langle D,V\rangle$ if α and β are both true in $\langle D,V\rangle$ or if α and β are both false in $\langle D,V\rangle$, and otherwise false in $\langle D,V\rangle$.

A wff of the form $\forall x\alpha$ is true in $\langle D,V\rangle$ if for every interpretation V', which gives all variables except x

the same values as V does, α is true in $\langle D,V'\rangle$, and otherwise it is false in $\langle D,V\rangle$.

A wff of the form $\exists x\alpha$ is true in $\langle D,V\rangle$ if there is an interpretation V′, which gives all variables except x the same values as V does, and α is true in $\langle D,V'\rangle$, and otherwise it is false in $\langle D,V\rangle$.

A wff of the form $L\alpha$ is true in $\langle D,V\rangle$ if for every interpretation V′, α is true in $\langle D,V'\rangle$, and otherwise it is false in $\langle D,V\rangle$.

A wff of the form $M\alpha$ is true in $\langle D,V\rangle$ if there is an interpretation V′, such that α is true in $\langle D,V'\rangle$, and otherwise it is false in $\langle D,V\rangle$.

30. A wff is valid in D if, for every interpretation V it is true in $\langle D,V\rangle$.

A wff is satisfiable in D if, there is an interpretation V such that it is true in $\langle D,V\rangle$.

A wff is valid if it is valid in every domain D.

A wff is satisfiable if, there is a domain D such that it is satisfiable in D.

We can turn our language $\mathfrak{L}$ into a deductive system by giving axioms and deduction rules.

The deductive system is sound if one can only prove valid wff.

The deductive system is complete if one can prove all valid wff.

31. The semantic rules that we have just given make first-order S5 unsound, and equally in the second-order case.

In first-order modal S5, and *a fortiori* in second-order modal S5, we have the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi x \vdash \varphi x}{\vdash \varphi x, \sim\varphi x}\ \text{I}\sim}{\vdash \varphi x \vee \sim\varphi x}\ \text{I}\vee}{\vdash L(\varphi x \vee \sim\varphi x)}\ \text{IL}}{\vdash \exists y L(\varphi x \vee \sim\varphi y)}\ \text{I}\exists$$

The conclusion of this deduction is not valid in the semantics proposed in section VI.

Assume a domain D composed of two individuals 0 and 1. Let ω be a one-place predicate such that ω(0) = T and ω(1) = F. Let V be an interpretation which gives the value ω to φ and 1 to x, whatever values it gives to the other variables of $\mathfrak{L}$. The wff $\exists y L(\varphi x \vee {\sim}\varphi x)$ will be false in ⟨D,V⟩.

For let V″ be an interpretation such that V(φ) = ω, V(x) = 1 and V(y) = 0, and where it does not matter what values V″ gives to the other variables of $\mathfrak{L}$. We have successively:

V″(φx) = F

V″(φy) = T

V″(~φy) = F

V″(φx ∨ ~φy) = F

For any interpretation V‴, V‴(L(φx ∨ ~φy)) = F.

In particular, for every interpretation V′ which gives all other variables the same values as ⟨D,V⟩, V′(L(φx ∨ ~φy)) = F. So V(∃yL(φx ∨ ~φy)) = F.

32. In second-order modal S5 we have the following deduction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi x \vdash \varphi x}{\vdash \varphi x, {\sim}\varphi x}\ \text{I}{\sim}}{\vdash \varphi x \vee {\sim}\varphi x}\ \text{I}\vee}{\vdash L(\varphi x \vee {\sim}\varphi x)}\ \text{IL}}{\vdash \exists \psi L(\varphi x \vee {\sim}\psi x)}\ \text{I}\exists$$

The conclusion of this deduction is not satisfiable in the semantics proposed in section VI.

Assume a domain D and an interpretation V. Let a be an individual in the domain D. Let ω be a one-place predicate such that ω(a) = F. Let ω′ be a one-place predicate such that ω′(a) = T.

Then let ⟨D,V″⟩ be an interpretation which gives the value a to x, ω to φ and ω′ to ψ, whatever values it gives to the other variables of $\mathfrak{L}$. We have successively:

V″(φx) = F

$V''(\psi x) = T$

$V''(\sim\psi x) = F$

$V''(\varphi x \vee \sim\psi x) = F$

So for all interpretations V''', $V'''(L(\varphi x \vee \sim\psi x)) = F$

In particular, for every interpretation V' which gives all other variables the same values as $\langle D,V\rangle$, $V'(L(\varphi x \vee \sim\psi x)) = F$. So $V(\exists\psi L(\varphi x \vee \sim\psi x)) = F$.

The proof holds for every domain D and interpretation V.

33. The problem with the sematic theory presented in section VI lies in the fact that it treats the symbols *L* and *M* as abbreviations for universal and existential closures. So that in expressions of the form $\exists y L(\varphi x \vee \sim\varphi y)$ or $\exists\psi L(\varphi x \vee \sim\psi x)$ the variables *y* and ψ are considered to be bound by *L* and *M* and not by the quantifiers $\exists y$ or $\exists\psi$, as they are in modal logic. Modal logic does not treat *L* and *M* as abbreviations for universal or existential closure. In other words modal logic does not identify the notions of validity and necessity.[22]

A. BAYART. *(Brussels).*

4. Commentary on CLM[23]

In CLM, 3 it should be noted that an interpretation V assigns values to the predicate and propositional variables, as well as to the individual variables. This is made plausible by Bayart's treatment of propositional and predicate symbols, as well as individual symbols, as variables which can all be bound by quantifiers. So we do not have the distinction, common in current treatments of first-order logic, between an interpretation to the predicates and an assignment to the individual variables. The use of 'propositional variable' and 'predicate variable', even for symbols never bound by quantifiers, as in propositional and first-order logic, was quite common at the time. It occurs in the contemporaneous Kripke 1959. The use of the interpretation V to give values to individual variables occurs also in Hughes and Cresswell 1968 (though not in Hughes and Cresswell 1996) where V in an LPC model assigns all values, alike to the predicate symbols and to the individual symbols.

A word of explanation needs to be said about the auxiliary language $\mathfrak{L}'$ introduced in CLM, II. It is important to appreciate that $\mathfrak{L}'$ contains symbols not in the object language $\mathfrak{L}$, in particular the abstraction operator λ, which can be used to form complex predicate expressions. Although this is not said explicitly, the use of $\mathfrak{L}'$ is motivated by the fact that Bayart is producing a semantics for second-order modal logic, in which both propositional and predicate variables can occur in quantifiers.[24] To see what

[22] There follows a bibliography, which has been incorporated in the list of references to the present article.

[23] I shall follow Bayart 1959 in referring to Bayart 1958 as CLM, and I shall refer to Bayart 1959 as QA. Following Bayart, an expression like CLM, II will refer to section II of CLM, and an expression like CLM, 9 will refer to paragraph 9 of CLM — and analogously with QA.

[24] The use of λ is found in the higher-order completeness proof in Henkin 1950, which Bayart was familiar with. For a hint of some of the complexities of substitution rules in higher-order logic see Henkin 1953 and Church 1956 p. 289f.

the problem is look first at a principle of ordinary first-order logic. This is the principle which can be stated by the schema

(1) $\forall x\alpha \supset \beta$

where β (which can be written $\alpha[y/x]$) is just like α except in having free y wherever α has free x.[25] The simplest instances of (1) are wff like $\forall x\varphi x \supset \varphi y$ — what is true of all is true of each. But more complex instances are such wff as $\forall x\exists z\forall v(\varphi xz \supset \varphi vx) \supset \exists z\forall v(\varphi yz \supset \varphi vy)$, but *not* $\forall x\exists z\forall y(\varphi xz \supset \varphi yx) \supset \exists z\forall y(\varphi yz \supset \varphi yy)$, since in the latter y becomes bound in β in a place where x was free in α. So much is standard, and is not difficult to articulate in first-order logic, where the replacement for x in α to get β is just another individual variable. (There are to be sure formulations of first-order logic which contain complex terms made up by the use of individual constants or function symbols, but in CLM, 2 Bayart excludes these.)

In the case of second-order logic we can have instances like

(2) $\forall\varphi\forall x\exists y(\varphi xy \wedge \varphi yx) \supset (\forall x\exists y((\psi x \equiv \psi y) \wedge (\psi y \equiv \psi x))$

where the simple two place predicate φ has been replaced by a complex expression, in such a way that φxy becomes $\psi x \equiv \psi y$, and φyx becomes $\psi y \equiv \psi x$. Of course we must rule out cases like

(3) $\forall\psi\exists y(\psi x \equiv \varphi yy) \supset \exists y(\varphi xy \equiv \varphi yy)$[26]

in which φxy is substituted for ψx, since that would cause a variable free in the substituting formula to become bound as a result of the substitution.

What has happened in (2) is that we have replaced a simple two-place predicate variable φ by a complex expression, ensuring that the variables which follow φ in the antecedent are retained in the consequent. So Bayart uses Church's device of lambda abstraction to systematise this fact. For any individual variable x you can read $\lambda x\alpha$ as 'is an x such that α', and λxy as 'are an x and y such that α', and in (2) one can think of replacing the simple predicate φ by the complex predicate expression, $\lambda zv(\psi z \equiv \psi v)$, which reads 'are a z and a v such that z is ψ iff v is ψ'. One can then represent (2) as

(4) $\forall\varphi\forall x\exists y(\varphi xy \wedge \varphi yx) \supset (\forall x\exists y(\lambda zv(\psi z \equiv \psi v)xy \wedge \forall x\exists y(\lambda zv(\psi z \equiv \psi v)yx)$[27]

What is going on in (4) is that φ has been replaced in each case by the complex two-place predicate $\lambda zv(\psi z \equiv \psi v)$, and this predicate takes the arguments x and y in that order when it replaces φxy, and the arguments y and x in that order when it replaces φyx. In (4) λzv is what Bayart calls an *n-place individual abstractor* and $\lambda zv(\psi z \equiv \psi v)$ an n-place individual abstract (or an *n-place parapredicate*). In the present

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[25]In a Gentzen system of the kind Bayart is using in this work, (1) would be written

$$\frac{\beta,\Gamma \vdash \Lambda}{\forall x\alpha,\Gamma \vdash \Lambda}$$

[26]Bayart uses lower case letters for variables of all kinds as well as for complex expressions. I follow him when principles are stated to hold for variables of all kinds, but in examples like (2) and (3) I have used $\forall\varphi$ of $\forall\psi$ rather than $\forall x$ or $\forall v$.

[27]In this case z and v have been used to avoid confusion with x and y, though in fact that is not strictly necessary, since in $\lambda zv(\psi z \equiv \psi v)xy$ you can think of λzv as a variable-binding operator whose scope does not extend to xy, and so even if $\lambda zv(\psi z \equiv \psi v)xy$ had been written as $\lambda xy(\psi x \equiv \psi y)xy$ the scope of λxy would still not have extended to the final x and y.

example n = 2 and we have a two-place abstract.

Bayart's use of the expression 'parapredicate' is to signal that this expression is not an expression in the object language $\mathfrak{L}$. So what are we to say of

(5) $\lambda zv(\psi z \equiv \psi v)xy$?

(5) could be read: x and y are a z and v such that $\psi z \equiv \psi v$. The wff of $\mathfrak{L}$ that (5) designates is what Bayart calls the *resultant* of (5). It is what you get by taking $\psi z \equiv \psi v$ and replacing z by x and v by y, i.e., it would be $\psi x \equiv \psi y$. So $\lambda zv(\psi z \equiv \psi v)yx$ would be what you get by taking $\psi z \equiv \psi v$ and replacing z by y and v by x, i.e. it would be $\psi y \equiv \psi x$. This procedure would turn (4) into (2). To take care of problems raised by examples like (3), where a variable becomes bound when replacing one which is free, Bayart, in CLM, 7, defines what he calls a 'well-formed abstract'. (5) itself is what Bayart calls a 'simple primary paraformula'.[28] In CLM, 17 Bayart reminds us, at the end of the third paragraph, that the paraformulae are not themselves part of the deduction system — that it is their resultants which are.

Bayart also has propositional abstracts, and n-place predicate abstracts. A propositional instance of (1) would be

(6) $\forall p(p \equiv q) \supset ((r \wedge s) \equiv q)$

which could be written

(7) $\forall p(p \equiv q) \supset \lambda v(v \equiv q)(r \wedge s)$

and then the rules for obtaining the resultant of $\lambda v(v \equiv q)(r \wedge s)$ would give $(r \wedge s) \equiv q$.

There will be some occasions on which the resultant will have to be obtained in several stages. Thus for instance in the expression

(8) $\lambda\varphi\forall x(\varphi x \equiv \psi x)(\lambda y\chi yy)$

where $\lambda y\chi yy$ is the predicate argument of the predicate abstract $\lambda\varphi\forall x(\varphi x \equiv \psi x)$ we first obtain the 'intermediate resultant'

(9) $\forall x((\lambda y\chi yy)xx \equiv \psi x)$

by eliminating the predicate abstract, and then obtain the final resultant by eliminating the individual abstract to get

(10) $\forall x(\chi xx \equiv \psi x)$.

Theorems I-VIII in CLM, III should now be straightforward results about the semantics of λ-expressions. Among the things they do is establish the semantic equivalence of a paraformula and its resultant.

[28]Bayart's actual phrase is 'paraproposition', but recall that I am referring to Bayart's 'propositions' as formulae.

In CLM, 25 Bayart notes that the only abstracts required in first-order logic have the form $\lambda x\alpha$ where x is an individual variable, so that (1) can be written as

(11) $\forall x\alpha \supset \lambda x\alpha y$[29]

where $\lambda x\alpha y$ is a paraformula whose resultant, provided $\lambda x\alpha(y)$is well-formed, is simply α with free y replacing free x.

In a Gentzen formulation of logic, in place of deriving theorems which are single wff, one derives what are called *sequents*. The usual way of understanding a sequent is as a pair $\langle\Phi,\Delta\rangle$ where Φ and Δ are sets of wff. Where $\langle\Phi,\Delta\rangle$ is a theorem of the logic we can write $\Phi \vdash \Delta$. Bayart's treatment in CLM, IV follows Gentzen's original paper more closely, except of course for the addition of the modal rules. (See Szabo 1969, pp. 83-85.) In the first place Φ and Δ, which he writes as ä and ë, are not sets but finite sequences. (That is why he needs 'structural' rules which guarantee that the order of wff does not matter, and that the repetition of wff does not matter.[30]) In the second place the symbol $\vdash$, Gentzen's $\rightarrow$, which Bayart writes as I, is not a sign that a sequent is derivable, but is a sign which separates Φ from Δ. So that where Φ is $\alpha_1,\ldots, \alpha_n$ and Δ is $\beta_1,\ldots, \beta_m$ then the sequent $\alpha_1,\ldots, \alpha_n \vdash \beta_1,\ldots, \beta_m$ is the n + m + 1 termed sequence whose first n terms are $\alpha_1,\ldots, \alpha_n$, and whose next term is $\vdash$, and whose final m terms are $\beta_1,\ldots, \beta_m$. So that instead of letting $\Phi \vdash \Delta$ indicate that $\langle\Phi,\Delta\rangle$ is a derivable sequent Bayart is forced to say that $\Phi \vdash \Delta$ is derivable. (Of course with I in place of $\vdash$ this looks less strange that it seems with $\vdash$.) What I have done in the translation is amalgamate these two uses of $\vdash$, since, even if, in strictness, this involves some blurring of the use/mention distinction, it seems to me that no serious confusion arises. Again, readers can be referred to the versions on the website. Since Φ and Δ are finite, if Φ is $\alpha_1,\ldots, \alpha_n$ and Δ is $\beta_1,\ldots, \beta_m$ one can write $\alpha_1,\ldots,\alpha_n \vdash \beta_1,\ldots,\beta_m$, and one can also write such things as $\Phi,\alpha \vdash \Delta,\beta$. Sequents can be given a semantics which describes them as true or false. Bayart calls $\Phi \vdash \Delta$ true if either one of the wff in Φ is false or one of the wff in Δ is true, and false otherwise. With this interpretation $\alpha_1,\ldots,\alpha_n \vdash \beta_1,\ldots,\beta_m$ will be true iff $(\alpha_1 \wedge\ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m)$ is true, and so $\alpha_1,\ldots,\alpha_n \vdash \beta_1,\ldots,\beta_m$ is equivalent to $\vdash (\alpha_1 \wedge\ldots \wedge \alpha_n) \supset (\beta_1 \vee \ldots \vee \beta_m)$, where $\vdash \Delta$ indicates a sequent in which Φ is empty. While this use of $\vdash \Delta$ is common, $\Phi \vdash$ is less so, though Bayart, again following Gentzen, makes extensive use of it. It would have to mean that Φ yields nothing, so if we let $\varnothing$ denote the empty sequence then $\Phi \vdash$ would be an abbreviation for $\Phi \vdash \varnothing$, and this would be true iff either some α in Φ is false or some β in $\varnothing$ is true. Since there is no β in Φ this is equivalent to saying that not every α in Φ is true, and so can be written $\Phi \vdash \perp$. This enables $\alpha_1,\ldots, \alpha_n \vdash$ to be understood as $\vdash (\alpha_1 \wedge\ldots \wedge \alpha_n) \supset \perp$, or equivalently $\vdash {\sim}(\alpha_1 \wedge\ldots \wedge \alpha_n)$, and I have used $\Phi \vdash \perp$ rather than $\Phi \vdash$ in the translation.

The rules that define the axiomatic Gentzen-style system for S5 predicate logic consist of one axiom and twenty eight transformation rules, what Bayart calls 'Rules of deduction'.[31] I shall illustrate the rules by looking at one of the rules for quantification, since this will also shew how abstraction is used by

[29] Although not strictly necessary this expression could do with some bracketing to make its meaning clear. We could write $(\lambda x\alpha)y$ or $\lambda x\alpha(y)$, though Bayart writes $\lambda x(\alpha)y$. (In his notation Zx(p)a looks less odd than $\lambda x(\alpha)y$. At CLM, 5 he points out that his introduction of brackets is for ease of reading.)

[30] In my commentary on Bayart's Henkin completeness proof, I shall frequently speak as if the components in a sequent are simply sets of wff rather than sequences. In QA, 23 (except for ä and ë) Bayart uses the same style of variable for sequences and sets.

[31] I have used Bayart's names for these except in adopting the Russellian symbols in place of Bayart's Polish symbols. So for instance Bayart's CI and IC become $\supset$I and I$\supset$. I have not replaced 'I' by $\vdash$ in the names of these rules since I suggests 'introduction' and all Bayart's rules are introduction rules.

Bayart. The rule is ∀I, which is

$$\frac{\lambda x(\alpha)\theta,\Phi \vdash \Delta}{\forall x\alpha,\Phi \vdash \Delta}$$

where θ might be a complex expression of the same type as x, as for instance when x is φ and θ is $\lambda zv(\psi z \equiv \psi v)$. In the first-order case we have seen that (the resultant of) $\lambda x\alpha(y)$ is just $\alpha[y/x]$, so that in the first-order case ∀I is

$$\frac{\alpha[y/x],\Phi \vdash \Delta}{\forall x\alpha,\Phi \vdash \Delta}$$

I.e., whatever you can deduce from $\alpha[y/x]$ together with Φ you can deduce from $\forall x\alpha$, together with Φ.

As far as the modal rules are concerned Bayart relies on an axiomatisation of S5 which is not so popular nowadays because it does not generalise to other modal systems. For L the rules say first that if you can get something from α you can get it from $L\alpha$, which is the equivalent of the T axiom $L\alpha \supset \alpha$, and second that if you can get α from wff which are all fully modalised — in the sense that all of their atomic wff are in the scope of a modal operator, then you can get $L\alpha$ from the same wff.[32] The rules for M are analogous.

5. Bayart 1959[33]

QUASI-COMPLETENESS OF SECOND-ORDER S5 AND COMPLETENESS OF FIRST-ORDER S5

In the present article we frequently refer to our earlier article 'La correction de la logique modale du premier et second ordre S5' (*Logique et Analyse*, 1). We refer to this as 'CLM'.

This article contains six sections and thirty three paragraphs. The references will take the form 'CLM, IV' or 'CLM, 12', referring respectively to the fourth section and to the twelfth paragraph.

I Quasi-semantic definitions for second-order logic

The language of second-order modal logic includes all the wff of second-order non-modal logic; these are the wff of the language $\mathfrak{L}$ defined in CLM, 2 which do not contain modal symbols. If the second-order modal logic defined in CLM, IV is complete in the sense defined in CLM, 4 it follows that all the wff valid in non-modal second-order logic will be derivable in second-order S5.

Now the set of derivable wff in second-order S5 is clearly recursively enumerable. In particular

[32]This axiomatisation is derives from A.N. Prior. See Prior 1955 pp. 202 and 306f, where Prior dates it from 1953. Prior's formulation in 1955, p. 202, is slightly defective, and, as noted in footnote 2 on p. 347 of Lemmon 1956, was corrected by Prior. Lemmon proves that the axiomatisation gives precisely S5 as formulated by Lewis. These facts are noted on p. 121 of Feys 1965 (added posthumously by Dopp).

[33]Translation of 'Quasi-adéquation de la logique modale de second ordre S5 et adéquation de la logique modale de premier ordre S5', *Logique et Analyse*, 2, 1959, 99–121.

the set of non-modal wff is recursively enumerable. But from Gödel's incompleteness theorem it follows that the set of valid formulae of non-modal second-order logic is not recursively enumerable. We must conclude that second-order S5 (which we shall call $S5^2$) cannot be complete.

This impossibility does not exist for first-order S5 and we shall prove the completeness of this logic.

All the same Henkin has shewn that non-modal second-order logic is complete in an extended sense which we may call 'quasi-complete'. We prove that $S5^2$ is quasi-complete in an analogous sense. In effect our exposition is no more than Henkin's theorem adapted for S5.

1. Let ⟨D,W⟩ be a universe composed of a set D of individuals and a set W of worlds and let *a* and *b* be the cardinal numbers of D and W respectively. In CLM, 1 we assumed, for each natural number n, a number $c = 2^{ba^n}$ of n-place intensional relations.

Assume, for each natural number n, a non-empty set P_n of n-place intensional relations based on ⟨D,W⟩. The sets D, W, P_0, P_1, P_2, ... based on ⟨D,W⟩ constitute a quasi-universe ⟨D,W,Q⟩ based on ⟨D,W⟩.

If for every natural number n, P_n contains all the n-place intensional relations in ⟨D,W⟩, ⟨D,W,Q⟩ will be a complete quasi-universe based on ⟨D,W⟩. In such a case we say that all the intensional relations in ⟨D,W⟩ are equally relative to ⟨D,W,Q⟩.

2. We take a second-order modal language $\mathfrak{L}$ defined as in CLM, 2. Consider a quasi-universe ⟨D,W,Q⟩ composed of the set D of individuals and W of worlds and sets of intensional relations P_0, P_1, P_2, ... We agree that the variables for individuals of the language $\mathfrak{L}$ take as their values the individuals of the set D and that for each natural number n the variables for n-place predicates take as their values the intensional relations in P_n.

If, in accordance with this convention, we are given a value to each of the variables of $\mathfrak{L}$ we are given an interpretation V relative to the quasi-universe ⟨D,W,Q⟩.

3. We take a quasi-universe ⟨D,W,Q⟩, a world *w* of this universe and an interpretation V relative to this universe. We then define the notions 'true for quasi-universe ⟨D,W,Q⟩, world *w* and interpretation V', and 'false for quasi-universe ⟨D,W,Q⟩, the world *w* and the interpretation V'.[34]

Let α be a wff of language $\mathfrak{L}$.

If α is a variable *p* for 0-place predicates, then if ω is the 0-place intensional relation given as the value of *p*, $V(\alpha,w) = \omega(w)$.

If α is $\varphi x_1...x_n$, where φ is an n-place predicate variable ($n \neq 0$) and where $x_1,...,x_n$ are individual variables, if ω, $a_1,..., a_n$ are respectively the n-place intensional relation and the individuals given as values of φ, $x_1,..., x_n$, $V(\alpha,w) = \omega(w,a_1, ..., a_n)$.

If α has the form ~β, where β is a wff, V(α,*w*) = T if V(β,*w*) = F, and V(α,*w*) = F if V(β,*w*) = T.

If α has the form $\beta \wedge \gamma$, where β and γ are wff, V(α,*w*) = T if V(β,*w*) = V(γ,*w*) = T, and V(α,*w*) = F otherwise.

If α has the form $\beta \vee \gamma$, where β and γ are wff, V(α,*w*) = T if V(β,*w*) = T or if V(γ,*w*) = T, and V(α,*w*) = F otherwise.

[34]Bayart uses 'quasi-true' and 'quasi-false' in line with his terminology for the sense of validity used to prove higher-order completeness in Henkin 1950, but, while I have retained such phrases as 'quasi-valid', 'quasi-universe', 'quasi-complete' and so on, I have simply used 'true' and 'false' or T and F, since, once an quasi-universe is given, *with respect to that quasi-universe* the notion of truth is so defined as to constrain the range within which higher-order quantifiers are evaluated, since in quasi-universes the higher-order variables must be given values from the allowable relations taken from P_0, P_1 etc. (This does mean that the truth clauses as given in this translation are almost verbatim repetitions of the truth definitions in CLM.)

If α has the form $\beta \supset \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta,w) = F$ or if $V(\gamma,w) = T$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\beta \equiv \gamma$, where β and γ are wff, $V(\alpha,w) = T$ if $V(\beta,w) = V(\gamma,w)$, and $V(\alpha,w) = F$ otherwise.

If α has the form $\forall x\beta$, where β is a wff and x a variable for individuals or predicates, $V(\alpha,w) = T$ if, for each interpretation V' relative to $\langle D,W,Q\rangle$ which gives to all the variables other than x the same values as V, $V'(\beta,w) = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $\exists x\beta$, where β is a wff and x a variable for individuals or predicates, $V(\alpha,w) = T$ if there is an interpretation V' relative to $\langle D,W,Q\rangle$ which gives to all the variables other than x the same values as V, and $V'(\beta,w) = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $L\beta$, where β is a wff, $V(\alpha,w) = T$ if for every world w' of the quasi-universe $\langle D,W,Q\rangle$, $V(\beta,w') = T$. Otherwise $V(\alpha,w) = F$.

If α has the form $M\beta$, where β is a wff, $V(\alpha,w) = T$ if there is a world w' of the quasi-universe $\langle D,W,Q\rangle$ such that $V(\beta,w') = T$, and otherwise $V(\alpha,w) = F$.

4. We take a quasi-universe $\langle D,W,Q\rangle$ and a world w of this universe. We define for formulae of the language $\mathfrak{L}$ the notions 'valid in $\langle\langle D,W,Q\rangle,w\rangle$' and 'satisfiable in $\langle\langle D,W,Q\rangle,w\rangle$'. Let α be a wff of $\mathfrak{L}$.

The wff α will be valid in $\langle\langle D,W,Q\rangle,w\rangle$ if and only if, for each interpretation V relative to $\langle D,W,Q\rangle$, $V(\alpha,w) = T$.

The wff α will be satisfiable in $\langle\langle D,W,Q\rangle,w\rangle$ iff there is an interpretation V relative to $\langle D,W,Q\rangle$ such that $V(\alpha,w) = T$.

The wff α will be valid in $\langle D,W,Q\rangle$ iff it is valid in every $\langle\langle D,W,Q\rangle,w\rangle$ (for every world w).

The wff α will be satisfiable in $\langle D,W,Q\rangle$ iff there is some world w such that α is satisfiable in $\langle\langle D,W,Q\rangle,w\rangle$.

We define for the language $\mathfrak{L}$ the notions 'quasi-valid' and 'quasi-satisfiable'.

The wff α will be quasi-valid iff it is valid in all quasi-universes.

The wff α will be quasi-satisfiable iff it is satisfiable in some quasi-universe.

We can express $\mathfrak{L}$ in a deductive system D_{S5} by being given axioms and rules of deduction. Assume a quasi-universe $\langle D,W,Q\rangle$.

The deductive system D_{S5} is quasi-sound for $\langle D,W,Q\rangle$ if one can only prove in D_{S5} formulae which are valid in $\langle D,W,Q\rangle$.

The deductive system D_{S5} is quasi-complete for $\langle D,W,Q\rangle$ if one can prove in D_{S5} all formulae which are valid in $\langle D,W,Q\rangle$.

5. It is easy to check that $S5^2$ is not sound with respect to every quasi-universe. Consider for instance a quasi-universe which for 0-place intensional relations contains only the function which takes the value F at every world. In $S5^2$ one can easily deduce the sequent $\vdash \exists pp$, where p is a propositional variable. But $\exists pp$ is not satisfiable in the present quasi-universe. So, to develop the quasi-soundness of $S5^2$ we must invoke the notion of a 'regular quasi-universe' as follows.

In CLM, 9 we gave a semantic definition of the value of an n-place individual abstract. We must now give the definition of the value of a wff for a universe $\langle D,W\rangle$ and an interpretation V. Let α be a wff of $\mathfrak{L}$. The value of α for V is the 0-place intensional relation ω such that for every world w of W, $\omega(w) = V(\alpha,w)$.

We now give the following quasi-semantical definitions for a quasi-universe $\langle D,W,Q\rangle$ based on an interpretation V relative to $\langle D,W,Q\rangle$.

The value of a wff α for ⟨D,W,Q⟩ and an interpretation V is the 0-place intensional relation which, for any world w of W, takes the value T or F according as $V(\alpha,w)$ = T of F.[35]

The value of an n-place individual abstract $\lambda x_1...x_n(\alpha)$ for a quasi-universe ⟨D,W,Q⟩ and an interpretation V relative to ⟨D,W⟩ is the n-place intensional relation which, for every world w and any series of individuals a_1, ..., a_n, takes the value T or F according as $V'(\alpha,w)$ = T or F, where V′ is an interpretation which assigns individuals a_1,..., a_n as values to the individual variables x_1,..., x_n respectively and which gives to all other variables the same values as V.

It is easy to see that the value of a wff or of an individual abstract is not always an intensional relation relative to ⟨D,W,Q⟩. Thus, in the quasi-universe described above the propositional variable p can only take a single value, and in the given value-system the value of $\sim p$ is not relative to ⟨D,W,Q⟩.

A quasi-universe ⟨D,W,Q⟩ is regular if, for every wff α of the language $\mathfrak{L}$, for every individual abstract $\lambda x_1...x_n(\alpha)$ constructed in the language $\mathfrak{L}$, and for every interpretation V relative to ⟨D,W,Q⟩, the value of α and the value of $\lambda x_1...x_n(\alpha)$ is an intensional relation relative to ⟨D,W,Q⟩.

It is clear that regular quasi-universes exist, notably the complete quasi-universes. The present exposition will shew that there also exist regular incomplete quasi-universes.

6. We can now present the series of our quasi-semantical definitions:

A wff is quasi-valid if and only if it is quasi-valid in all regular quasi-universes.

A wff is quasi-satisfiable if and only if there is a regular quasi-universe in which it is quasi-satisfiable.

A deductive system D_{S5} is quasi-sound if all wff derivable in D_{S5} are quasi-valid.

A deductive system D_{S5} is quasi-complete if one can prove in D_{S5} all formulae which are quasi-valid.

II Sematic properties of paraformulae

7. In what follows we adapt the semantic theorems of CLM, III. Certain of the quasi-semantical theorems which follow hold for every quasi-universe, others only hold for regular quasi-universes. We will indicate each time which of these is the case.

8. *Theorem I*: Consider a universe ⟨D,W,Q⟩, two worlds w and w' of W and any interpretation relative to ⟨D,W,Q⟩. If α is a modalised wff then $V(\alpha,w) = V(\alpha,w')$.

9. *Theorem II*: Let α be a wff containing only x_1,..., x_n as free variables. Consider any universe ⟨D,W,Q⟩, a world w of W and two interpretations V and V′ relative to ⟨D,W,Q⟩ which do not differ in the values assigned to x_1,..., x_n. Then $V(\alpha,w) = V'(\alpha,w)$. In particular if α is a closed wff (i.e., does not contain free variables) then for any two interpretations V and V′ relative to ⟨D,W,Q⟩, $V(\alpha,w) = V'(\alpha,w)$.

10. *Theorem III*: Let α be a wff containing only x_1,..., x_n as free variables. Consider any universe ⟨D,W,Q⟩, and two interpretations V and V′ relative to ⟨D,W,Q⟩ which do not differ in the values assigned to x_1,..., x_n. Then $V(\alpha) = V'(\alpha)$. (For $V(\alpha)$ see paragraph 5 above.) In particular if α is a closed proposition

[35] This looks like a repetition of what was said two paragraphs earlier. But Bayart is now defining truth and falsity in a *quasi-universe*. (If we were to follow Bayart's use of 'quasi-true' and 'quasi-false' the impression of repetitiveness would disappear.)

then for any two interpretations V and V′ relative to ⟨D,W,Q⟩, V(α) = V′(α).

We could have formulated a semantic analogue of theorem III in CLM, III.

11. *Theorem IV*: Let θ be an individual abstract $\lambda x_1 ... x_n(\alpha)$ which contains only the variables $y_1,..., y_n$ free. Take any universe ⟨D,W,Q⟩ and any two interpretations V and V′ relative to ⟨D,W,Q⟩ which do not differ in the values given to the variables $y_1,..., y_n$. Then V(θ) = V′(θ). In particular if θ is a closed abstract then for any two interpretations V and V′ relative to ⟨D,W,Q⟩, V(θ) = V′(θ).

The value of the wff α in theorem III and that of the abstract θ in theorem IV are values relative to ⟨D,W⟩ and not necessarily values relative to ⟨D,W,Q⟩.

12. *Theorem V*: For any universe ⟨D,W,Q⟩, and world *w* of W and any interpretation V relative to ⟨D,W,Q⟩, if $\lambda x_1 ... x_n(\alpha) y_1 ... y_n$ is a well-formed simple primary paraformula and α′ is the resultant of this formula then V(α′,*w*) = V′(α,*w*), where V′ is an interpretation which gives to the individual variables $x_1,...,x_n$ the individuals $a_1,..., a_n$ respectively, being the same individuals as assigned by V to the variables $y_1,..., y_n$ respectively, and which gives all other variables the same values as V does.

13. *Theorem VI*: For any universe ⟨D,W,Q⟩, any world *w* of W, and any interpretation V relative to ⟨D,W,Q⟩, if $\lambda x_1 ... x_n(\alpha) y_1 ... y_n$ is a well-formed simple primary paraformula, and if ω is the n-place intensional relation which is the value given by V to the individual abstract $\lambda x_1 ... x_n(\alpha)$, the value V(α′,*w*) of the resultant α′ of this paraformula will be $\omega(w, a_1,..., a_n)$ where $a_1,..., a_n$ are the values given by V to the variables $y_1,..., y_n$.

The relation ω relative to ⟨D,W⟩ is not necessarily relative to ⟨D,W,Q⟩.

14. *Theorem VII*: For any regular quasi-universe ⟨D,W,Q⟩, any world *w* of ⟨D,W,Q⟩, and any interpretation V relative to ⟨D,W,Q⟩, if λ*p*(α)β is a well-formed propositional paraformula, and α′ is the resultant of this paraformula then V(α′,*w*) = V′(α,*w*) where V′ is the interpretation which assigns the propositional variable *p* the 0-place relation ω such that ω = V(β), and which gives all the other variables the same value as V.

(The analogous theorem VI of CLM, 15 could have been stated as follows: For any universe ⟨D,W⟩, any world *w* of ⟨D,W⟩, and any interpretation V relative to ⟨D,W⟩, if λφ(α)θ is a well-formed propositional paraformula, where φ is an n-place predicate variable and θ is an n-place individual abstract, the value given by V in *w* of the final resultant α′ of this paraformula is the same as V′(α), where V′ is the interpretation which assigns to the propositional variable *p* the 0-place relation ω such that ω = V(β), and which gives all the other variables the same value as V.)

15. *Theorem VIII*: For any regular quasi-universe ⟨D,W,Q⟩, any world *w* of W, and any interpretation V relative to ⟨D,W,Q⟩, if λφ(α)θ is a well-formed predicate paraformula, where φ is an n-place predicate variable and θ is an n-place individual abstract, the value in *w* of the final resultant V(α′) = V′(α) where V′ is the interpretation which assigns to the variable φ the value V(θ) and which gives all the other variables the same values as V.

In theorems VII and VIII, from the fact that ⟨D,W,Q⟩ is a regular quasi-universe, the intensional relation ω is relative to ⟨D,W,Q⟩, and so it is possible to use the interpretation V′ described in these theorems.

16. *Theorem IX*: Let α be a wff. Let *x* be a variable. Let *y* be a variable of the same type as *x* which does not occur, either free or bound, in α.

Let β be the wff obtained by substituting in α the variable *y* for the variable *x* wherever the latter occurs bound (β being identical with α if *x* is not bound in α.) Then, for any quasi-universe ⟨D,W,Q⟩, any world *w* and any interpretation V relative to ⟨D,W,Q⟩, α and β have the same value in *w*.

Proof by induction on the construction of α, distinguishing between cases where α has the form ∀*x*α or ∃*x*α, and those where α has the form ∀*z*α or ∃*z*α, *z* being a variable distinct from *x* and *y*.

In CLM, III we could have formulated a sematic theory analogous to the present theorem IX, but such a theorem is not needed.

III Quasi-soundness and quasi-completeness of $S5^2$

17. We say that a wff α is derivable in $S5^2$ if the sequent ├ α is derivable in $S5^2$.

We say that a sequent Φ ├ Δ is true in *w* (for a quasi-universe ⟨D,W,Q⟩ and an interpretation V) if Φ contains a wff false in *w* or if Δ contains a wff true in *w*. Otherwise the sequent Φ ├ Δ is false in *w*.

One can then easily define quasi-validity and quasi-satisfaction for sequents.

We say that the wff α represents the sequent Φ ├ Δ if α is a disjunction whose disjuncts, in order, are the negations of the wff in Φ followed by the wff in Δ. One can easily shew that Φ ├ Δ is derivable in $S5^2$ iff α is derivable in $S5^2$.

One can equally easily shew that Φ ├ Δ is true or false in *w*, iff α is true or false in *w*.

It follows that the quasi-soundness and quasi-completeness of $S5^2$ can be equally defined in terms of wff or in terms of sequents.

18. *Theorem X*: If all wff derivable in $S5^2$ are quasi-satisfiable in a quasi-universe, then all wff derivable in $S5^2$ are quasi-valid in ⟨D,W,Q⟩.

Proof from the fact that if a wff α is derivable in $S5^2$ the wff *L*∀α is equally so. ∀α designates here the universal closure of α.

19. *Theorem XI*:If $S5^2$ is quasi-sound for a quasi-universe ⟨D,W,Q⟩, ⟨D,W,Q⟩ is a regular quasi-universe.

Proof: From the definitions of a quasi-sound system and a regular quasi-universe, and from the fact that all wff of the form $\exists\varphi L\forall x_1...\forall x_n(\varphi x_1...x_n \equiv \alpha)$, where φ is an n-place predicate variable, and where x_1,..., x_n are n distinct individual variables, and where β is a wff not containing free φ, and thus all wff of the form $\exists pL(p \equiv \alpha)$ where *p* is a propositional variable, and where β is a wff not containing free *p*, are derivable in $S5^2$.

20 *Theorem XII*: $S5^2$ is quasi-sound

The proof is analogous to the proof of the soundness of $S5^2$, given in CLM, IV. It must take account of the fact that quasi-soundness has been defined in paragraph 6 above in terms of regular quasi-universes.

The soundness proof for ∀I (see CLM, 21) is based on the quasi-semantical theorems V, VII or VIII. Because the universes considered are regular it is possible to provide an interpretation V which gives to the variable x the value given by V to the argument θ of the paraformula $\lambda x(\alpha)\theta$.

21 *Theorem XIII*: If α is a consistent wff, i.e., if the sequent $\alpha \vdash \perp$ is not derivable in $S5^2$, α is quasi-satisfiable.

Proof: Section IV of the present article will establish, for every consistent wff α, a regular quasi-universe $\langle D,W,Q \rangle$ such that α is satisfiable in $\langle D,W,Q \rangle$.

22. *Theorem XIV*: $S5^2$ is quasi-complete

Proof: If α is quasi-valid, $\sim\alpha$ will be a wff which is not quasi-satisfiable. By contraposition of theorem XIII we obtain that the sequent $\sim\alpha \vdash \perp$ is derivable, from which it easily follows that the sequent $\vdash \alpha$ is derivable.

IV Proof of theorem XIII

23. In what follows we understand by 'well-formed formula' (wff) a wff of language $\mathfrak{L}$ defined in CLM, 2 and by 'wff or derivable sequent' we mean a wff or sequent derivable in $S5^2$.

We use α, β, γ etc. to designate wff. These letters may be followed by one or two numerical indices.

The capital letters B, D, F etc., and Greek letters like Λ, Φ, Δ etc., designate series or finite or infinite sets of wff. These expressions may be followed by one or two numerical indices.

Use of these syntactical notations may be combined with the preceding syntactical notations.

If all the wff of a set or series B of wff are elements of a set E of wff we say that the set or series B is drawn from the set E.

24. A finite or infinite set B of wff is consistent if there is no finite series Φ included in B such that $\Phi \vdash \perp$ is derivable.

A finite or infinite series of wff is consistent if it is included in a consistent set.

A wff α is consistent with a set B of wff if $B \cup \{\alpha\}$ is consistent.

It is easy to shew that if Φ is a finite series of wff included in a consistent set B, and if $\Phi \vdash \alpha$ is derivable then α is consistent with B. *A fortiori*, if $\vdash \alpha$ is provable it is consistent with every consistent set.

25. Let α^* be a consistent wff. We order the set of wff of the form $M\beta$ in a series $M\beta_0, M\beta_1, M\beta_2, \ldots$ We order the set of wff of the form $\exists x\delta$ where x is any variable in a series $\exists x_1\delta_1, \exists x_2\delta_2, \exists x_3\delta_3, \ldots$

Consider the set of ordered pairs of natural numbers and order it diagonally as follows: 00, 01, 10, 11, 20,03, ... Assume the following series of wff $\gamma_{0.0}, \gamma_{0.1}, \gamma_{1.0} \ldots$

For each natural number n, $\gamma_{n.0}$ is the wff $M\alpha^* \wedge (M\beta \supset \beta)$ where $M\beta = M\beta_n$.

For each pair of natural numbers n and m such that $m \neq 0$, $\gamma_{n.m}$ is the wff $\exists x_m\delta_m \supset \delta_m[y/x_m]$ where y designates the first variable in alphabetical order of the same type as x which does not occur free in $\exists x\delta$ nor in any wff $\gamma_{r.s}$ where r.s is an index which precedes 'n.m'.

We assume the following set of wff $\zeta_{0.0}$, $\zeta_{0.1}$, $\zeta_{1.0}$ ... For each natural number n, ζ_{n0} is $M\gamma_{n.0}$. For each pair of natural numbers n and m such that m $\neq$ 0, $\zeta_{n.m}$ is the wff $M(\gamma_{n.0} \wedge ... \wedge \gamma_{n.m})$.

26. Consider the set G of wff $\zeta_{0.0}$, $\zeta_{0.1}$, $\zeta_{1.0}$ ...

Lemma I The set G as defined above is consistent

Proof by reductio. Let Λ be a finite series included in G such that $\Lambda \vdash \bot$ is derivable. Let $\zeta_{n.m}$ be the wff of Λ such that no other wff of Λ has an index of higher rank than n.m. let Λ' be the series composed of all the wff $\zeta_{r.s}$ appearing or not in Λ whose index is lower than n.m. It is clear that if $\Lambda \vdash \bot$ is derivable then $\zeta_{n.m},\Lambda' \vdash \bot$ is also.

We shew that the latter is impossible by induction on the rank of the index n.m.

Suppose n = m = 0. Then $\zeta_{0.0}$ is a wff of the form $M(M\alpha^* \wedge (M\beta \supset \beta))$ and Λ is empty. We then suppose that $M(M\alpha^* \wedge (M\beta \supset \beta)) \vdash \bot$ is provable. As we have $M\alpha^* \wedge (M\beta \supset \beta) \vdash M(M\alpha^* \wedge (M\beta \supset \beta))$ we obtain by a cut that $M\alpha^* \wedge (M\beta \supset \beta) \vdash \bot$ is derivable. As we have $M\alpha^*,M\beta \supset \beta \vdash M\alpha^* \wedge (M\beta \supset \beta)$ we obtain by a cut that $M\alpha^*,(M\beta \supset \beta) \vdash \bot$ is derivable. Since $M\alpha^*$ is modalised we have that $M(M\beta \supset \beta),M\alpha^* \vdash \bot$ is derivable.

But $\vdash M(M\beta \supset \beta)$ is derivable as follows:

$$\dfrac{\dfrac{\dfrac{M\beta \vdash \beta, M\beta}{\vdash M\beta \supset \beta, M\beta}}{\vdash M(M\beta \supset \beta), M\beta} \qquad \dfrac{\dfrac{\dfrac{M\beta,\beta \vdash \beta}{\beta \vdash M\beta \supset \beta}}{\beta \vdash M(M\beta \supset \beta)}}{M\beta \vdash M(M\beta \supset \beta)}}{\vdash M(M\beta \supset \beta)}$$

Hence by a cut with $M(M\beta \supset \beta),M\alpha^* \vdash \bot$ we obtain that $M\alpha^* \vdash \bot$ is derivable, contrary to the hypothesis according to which it is a consistent wff.

Suppose n $\neq$ 0 and m = 0. $\zeta_{n.m}$ then has the form $M(M\alpha^* \wedge (M\beta \supset \beta))$ but Λ' is no longer empty.

Suppose then that $M(M\alpha^* \wedge (M\beta \supset \beta)),\Lambda' \vdash \bot$ is derivable. We deduce successively that the following sequents are derivable:

$M\alpha^* \wedge (M\beta \supset \beta),\Lambda' \vdash \bot$
$M\alpha^*,M\beta \supset \beta,\Lambda' \vdash \bot$
$M(M\beta \supset \beta),M\alpha^*,\Lambda' \vdash \bot$ (since all the wff in Λ are modalised.)
$M\alpha^*,\Lambda' \vdash \bot$ (since $\vdash M(M\beta \supset \beta)$ is derivable.)

But Λ contains the wff $\zeta_{0.0}$ which has the form $M(M\alpha^* \wedge (M\beta' \supset \beta'))$. Call this wff '$\beta^*$'. Now we have the following proof:

$$\frac{\dfrac{M\alpha^*, M\beta' \supset \beta' \vdash M\alpha^*}{M\alpha^* \wedge (M\beta' \supset \beta') \vdash M\alpha^*}}{M(M\alpha^* \wedge (M\beta' \supset \beta')) \vdash M\alpha^*}$$

I.e., that $\beta^* \vdash M\alpha^*$ is derivable, whence by a cut with $M\alpha^*,\Lambda' \vdash \bot$ we obtain $\beta^*,\Lambda' \vdash \bot$.

But β^* is a wff of Λ'. Thus we have $\Lambda' \vdash \bot$ contrary to the induction hypothesis.

Suppose n is any number and m ≠ 0. Then $\zeta_{n.m}$ has the form $M(\gamma_0 \wedge ... \wedge \gamma_m)$ where γ_m has the form $\exists x\delta \supset \delta[y/x]$. We then suppose that $M(\gamma_0 \wedge ... \wedge \alpha_m),\Lambda' \vdash \bot$ is derivable. As $\zeta_{n.m}$ has an index of higher rank than all the other wff of Λ', and as γ_m is the wff $\gamma_{n.m}$ of which the index is of greater rank than all the other wff which enter into the composition of $\zeta_{n.m}$[36] or of a wff of Λ', we have that the variable y does not occur free or bound except in $\delta[y/x]$.

Hence, if $M(\gamma_0 \wedge ... \wedge \gamma_m),\Lambda' \vdash \bot$ is derivable, the following sequents are also:

$(\gamma_0 \wedge ... \wedge \gamma_m),\Lambda' \vdash \bot$

$(\gamma_0 \wedge ... \wedge \gamma_{m-1}),\gamma_m,\Lambda' \vdash \bot$ or, what amounts to the same

$(\gamma_0 \wedge ... \wedge \gamma_{m-1}),\exists x\delta \supset \delta[y/x],\Lambda' \vdash \bot$

$\exists y(\exists x\delta \supset \delta[y/x]),(\gamma_0 \wedge ... \wedge \gamma_{m-1}),\Lambda' \vdash \bot$

(in virtue of what has been said about the variable y.)

But $\vdash \exists y(\exists x\delta \supset \delta[y/x])$ is derivable as follows:

$$\frac{\dfrac{\dfrac{\exists x\delta \vdash \delta[y/x],\exists x\delta}{\vdash \exists x\delta \supset \delta[y/x],\exists x\delta}}{\vdash \exists y(\exists x\delta \supset \delta[y/x]),\exists x\delta}\,(3) \qquad \dfrac{\dfrac{\dfrac{\exists x\delta,\delta \vdash \delta}{\delta \vdash \exists x\delta \supset \delta}}{\delta \vdash \exists y(\exists x\delta \supset \delta[y/x])}\,(1)}{\exists x\delta \vdash \exists y(\exists x\delta \supset \delta[y/x])}\,(2)}{\vdash \exists y(\exists x\delta \supset \delta[y/x])}$$

[36] $\zeta_{n.m}$ will have the form $M(\gamma_{n.0} \wedge ... \wedge \gamma_{n.m})$, and Bayart means by his sentence that all the indices of these γs other than $\gamma_{n.m}$ are lower than n.m.

To enable verification of the legitimacy of this proof it is pointful to make the following remarks

(1) $\delta[y/x] = \lambda x(\delta)y$ where y does not occur in δ. It follows from this that $\delta = \lambda y(\delta[y/x])x$ and that $\exists x\delta \supset \delta = \lambda y(\exists x\delta \supset \delta[y/x])x$

(2) The variable y does not occur free in $\exists y(\exists x\delta \supset \delta[y/x])$.

(3) $\exists x\delta \supset \delta[y/x] = \lambda y(\exists x\delta \supset \delta[y/x])y$

From $\exists y(\exists x\delta \supset \delta[y/x]),\gamma_0 \wedge ... \wedge \gamma_{m-1},\Lambda' \vdash \bot$ and from $\vdash \exists y(\exists x\delta \supset \delta[y/x])$ we obtain by a cut, $\gamma_0 \wedge ... \wedge \gamma_{m-1},\Lambda' \vdash \bot$. Noting that all the wff of Λ' are modalised we obtain $M(\gamma_0 \wedge ... \wedge \gamma_{m-1}),\Lambda' \vdash \bot$. But $M(\gamma_0 \wedge ... \wedge \gamma_{m-1}),\Lambda'$ is a wff of Λ'. Hence we obtain $\Lambda' \vdash \bot$ contrary to induction hypothesis. This completes the proof of the lemma.

27. Consider the set of all modalised wff and order this in a series η_1, η_2, η_3, ... We assume the following series of sets of wff H_0, H_1, H_2, ...

$H_0 = G$.

$H_{n+1} = H_n$ if the wff η_{n+1} is inconsistent with H_n and otherwise $H_{n+1} = H_n \cup \{\eta_{n+1}\}$

We see immediately by induction on n, and noting that G is consistent, that for every n, H_n is consistent.

Let H be the union of H_0, H_1, H_2,...

Lemma II: H is consistent

Proof by reductio. Let Λ be a series included in H such that $\Lambda \vdash \bot$ is derivable. Let η_n be the wff with the highest index in Λ. It is clear that all the wff of Λ appear in H_n. Then H_n will be inconsistent, contrary to construction.

Lemma III: If η is a modalised wff then if η is consistent with H then η is an element of H.

Proof: Let the index of η in the series η_1, η_2, η_3 be n. If η is consistent with H then it is consistent with H_{n-1}. From this we have by construction that $H_n = H_{n-1} \cup \{\eta\}$. So η is an element of H.

28. Assume the series F_0, F_1, F_2 containing respectively the wff $\gamma_{0.0}, \gamma_{0.1}, \gamma_{0.2}, ..., \gamma_{1.0}, \gamma_{1.1}, \gamma_{1.2}, ..., \gamma_{2.0}, \gamma_{2.1}, \gamma_{2.2}, ...,$

Assume the series Q_0, Q_1, Q_2, ... defined as follows: $Q_0 = H \cup F_0$; $Q_1 = H \cup F_1$; $Q_2 = H \cup F_2$, ...

Lemma IV: The sets Q_0, Q_1, Q_2, ... are consistent

Proof by reductio. Consider some series Q_n. Let Λ be a series included in Q_n such that $\Lambda \vdash \bot$ is derivable. Let Λ' be the series composed of those elements of Λ which are elements of F_n and let Λ'' be that which remains in the series Λ when all the elements of Λ' are removed. Let Λ''' be the series $\gamma_{n.0}, ..., \gamma_{n.m}$ where m is the highest number occurring in the second index of a wff in Λ'. It is clear that if $\Lambda \vdash \bot$ is derivable then $\Lambda''',\Lambda'' \vdash \bot$ is equally. Consider the wff $(\gamma_0 \wedge ... \wedge \gamma_m)$ where $\gamma_0,..., \gamma_m$ are respectively the wff $\gamma_{n.0}, ..., \gamma_{n.m}$. We would have that $(\gamma_0 \wedge ... \wedge \gamma_m),\Lambda'' \vdash \bot$ is derivable. Taking account of the fact that all the wff of Λ are elements of H and thus are modalised wff we would have that $M(\gamma_0 \wedge ... \wedge \gamma_m),\Lambda'' \vdash \bot$ is derivable. But $M(\gamma_0 \wedge ... \wedge \gamma_m) = \zeta_{n.m}$, and $\zeta_{n.m}$, like all the wff of Λ, is an element of H. It follows that H would be inconsistent, contrary to lemma II.

It is clear that identical reasoning holds equally for the case where Λ contains only the wff $\gamma_{n.0}$.

29. Consider the set of all wff and order them in a series $\alpha_1, \alpha_2, \alpha_3$ ... defined as follows:
For each number n $R_{n.0} = Q_{0n}$. For each number m+1 $R_{n.m+1} = R_{n.m}$ if α_{m+1} is inconsistent with $R_{n.m}$ and otherwise $R_{n.m+1} = R_{n.m} \cup \{\alpha_{m+1}\}$. We see immediately by induction on m, and considering that Q_n is consistent, that for each m $R_{n.m}$ is consistent.

Consider the sets R_0, R_1, R_2 ... which are respectively the unions of the sets $R_{0.0}, R_{0.1}, R_{0.2}$ ...$R_{1.0}$, $R_{1.1}, R_{1.2}$ ... $R_{2.0}, R_{2.1}, R_{2.2}$, ...

Lemma V: The sets R_0, R_1, R_2 ... are consistent.

Proof by reductio. Let Λ be a series included in R_n such that $\Lambda \vdash \perp$ is derivable. Let α_m be the wff of Λ whose index m is the highest. It is clear that all the wff of Λ appear in $R_{n.m}$. Hence $R_{n.m}$ is inconsistent, contrary to construction.

Lemma VI: Let α be a wff. If α is consistent with R_n α is an element of R_n.

Proof: Let the index of α in the series $\alpha_1, \alpha_2, \alpha_3$ be m. If α is consistent with R_n it is consistent with $R_{n.m+1}$. from this we have, by definition, that $R_{n.m} = R_{n.m-1} \cup \{\alpha\}$. So α is an element of R_n.

Lemma VII: If α is a modalised wff and if α appears in a set R_n then, for all m, it appears in R_m

Proof: Let i be the index of α in the series $\alpha_1, \alpha_2, \alpha_3$, ... If α belongs to R_n then α is consistent with $R_{n.i-1}$. But $R_{n.i-1}$ contains H. So α is an element of H. From this, in virtue of the manner of definition of the set R_0, R_1, R_2 ... α is an element of each of these sets.

30. Assume a quasi-universe $\langle D,W,Q \rangle$ containing a denumerably infinite set of individuals, and a denumerably infinite set of worlds.

Assume a 1-1 correspondence between individual variables and the individuals of $\langle D,W \rangle$.

Assume a 1-1 correspondence between the sets R_0, R_1, R_2 ... and the worlds of $\langle D,W \rangle$. Consider the set of intensional relations which are given by $\langle D,W \rangle$. For each natural number n we establish a correspondence between n-place predicate variables and certain n-place intensional relations such that to each variable corresponds a single predicate, though several variables may correspond to the same predicate.

If *p* is a propositional variable we let correspond to *p* the 0-place intensional relation ω which takes the value T for the worlds corresponding to the sets R_n which contain *p*, and the value F for the other worlds.

If φ is an n-place predicate variable ($n \neq 0$) we let correspond to φ the n-place intensional relation ω which, when given as arguments a world *w* and the individuals $a_1, ..., a_n$ (not necessarily distinct), takes the value T or F according as the wff $\varphi x_1 ... x_n$ is contained or not in the set R_w, the set R_w being that which corresponds to the world *w* and the variables $x_1,..., x_n$ being those which correspond to the individuals a_1, ..., a_n respectively.

Consider the set of intensional relations of $\langle D,W \rangle$ which we have made correspond with the variables of $\mathfrak{L}$. This set of predicates constitutes, with the set of individuals and the set of worlds of $\langle D,W \rangle$, a quasi-universe ⟨D,W,Q⟩ based on ⟨D,W⟩. Further, the system of correspondences established constitutes an interpretation V, relative to ⟨D,W,Q⟩. It is clear that the quasi-universe ⟨D,W,Q⟩ permits the establishing of other interpretations than V.

31. *Lemma VIII*: Let ⟨D,W,Q⟩ be a quasi-universe and V the interpretation relative to ⟨D,W,Q⟩ corresponding with the set R_w. Let α be a wff. Then α is true or false in *w* according as α occurs or not in R_w.

Proof by induction on the construction of α. (v. remarks at the end of the present paragraph.)

If α is an atomic wff the lemma follows from the correspondences established between the variables of $\mathfrak{L}$ and the quasi-universe ⟨D,W,Q⟩.

If α has the form ~β and if ~β is in R_w then β is not in R_w, for otherwise R_w would be inconsistent. So β is false in *w* and ~β is true in *w*.

If α, i.e. ~β, does not appear in R_w, then β appears in R_w, for if not it would follow that ~β and β are both inconsistent with R_w. We would then have the derivable sequents ~β,Λ ├ ⊥ and β,Λ′ ├ ⊥ where Λ and Λ′ are sequents taken from R_w. Let Λ″ = Λ ∪ Λ′. We then have ~β,Λ″ ├ ⊥ and β,Λ″ ├ ⊥ and easily obtain Λ″ ├ β. By a cut with β,Λ″ ├ ⊥ we obtain Λ″ ├ ⊥ and therefore that R_w is inconsistent. If β is in R_w, β is true in *w*, and so ~β is false.

If α has the form β ∧ γ and α appears in R_w, β and γ appear in R_w. For β ∧ γ ├ β and β ∧ γ ├ γ are derivable. So β and γ are consistent with R_w, and from this are clearly in R_w. So β and γ are true in *w*, and so β ∧ γ is true in *w*.

If α, i.e. β ∧ γ does not appear in R_w. β and γ cannot both appear, for otherwise, since the sequent β, γ ├ β ∧ γ is derivable, β ∧ γ would be in R_w. One of the two wff β and γ will not be in R_w, and this one will be false in *w*. So β ∧ γ is false in *w*.

If α has the form β ∨ γ and α appears in R_w, one of the wff β and γ will appear in R_w, for otherwise ~β and ~γ will appear, and since ~β,~γ,β ∨ γ ├ ⊥, R_w will be inconsistent. Whichever wff β or γ appears in R_w will be true, and so β ∨ γ will be true in *w*.

If α, i.e., β ∨ γ does not appear in R_w, then neither β nor γ appear in R_w. For otherwise, since β ├ β ∨ γ and γ ├ β ∨ γ are derivable β ∨ γ will appear in R_w. So β and γ are false in *w*, and from this β ∨ γ is false in *w*.

If α has the form β ⊃ γ and α appears in R_w, γ will appear in R_w or β will not be in R_w, for otherwise β and ~γ will appear, and since ~γ,β,β ⊃ γ ├ ⊥ is derivable, R_w will be inconsistent. If γ appears in R_w then γ will be true in *w*, and if β does not appear in R_w then β will be false in *w*, and in either case β ⊃ γ will be true in *w*.

If α, i.e., β ⊃ γ does not appear in R_w, then γ will not appear in R_w and β will appear in R_w. For otherwise, γ or ~β will be in R_w, and since γ ├ β ⊃ γ and ~β ├ β ⊃ γ are derivable β ⊃ γ will appear in R_w. So γ is false in *w* and β is true in *w*, and from this β ⊃ γ is false in *w*.

If α has the form β ≡ γ and α appears in R_w, β and γ will both be in R_w or neither β nor γ will be in R_w, For if one of these wff is in R_w and the other is not, one will have, for instance, that β and ~γ are in R_w. But ~γ,β,β ≡ γ ├ ⊥ is derivable. It follows that β and γ are both true in *w* or that β and γ are both false in *w*, and so β ≡ γ is true.

If α, i.e., β ≡ γ does not appear in R_w, then one of the wff β and γ will appear in R_w and the other not. For, if both wff appear then one notes that β,γ ├ β ≡ γ is derivable, and if neither β nor γ is in R_w then ~β and ~γ are in R_w, and ~β,~γ ├ β ≡ γ is derivable. So one of the two wff must be true in *w* and one false in w, and from this $\beta \equiv \gamma$ is false in w.

If α has the form $\forall x\beta$ and if α occurs in R_w, then for every interpretation V′ which gives all variables other than x the same value as V, β is true in w according to V′. For let ω be the entity (individual or relation) V′ makes correspond with the variable x, and let y be the variable, of the same type as x, which V makes correspond with ω. Two hypotheses arise according as $\lambda x(\beta)y$ is a well-formed paraformula or not.

If $\lambda x(\beta)y$ is well-formed let γ be its resultant. Then, since $\forall x\beta \vdash \gamma$ is derivable, γ, appears in R_w and is thus true in w. But in virtue of theorems V, VII or VIII, γ has, in w, the value which β has in w according to V′. Thus $V'(\beta,w) = T$.

If $\lambda x(\beta)y$ is not well-formed it will be because x occurs free in β in the scope of a quantifier $\forall y$ or $\exists y$. let β' be the wff obtained by replacing in β the variable y everywhere it occurs bound by a variable z of the same type which does not occur in $\forall x\beta$, hence not in β, free or bound. $\forall x\beta \vdash \forall x\beta'$ is derivable and hence $\forall x\beta'$ is an element of R_w. Further $\lambda x(\beta')y$ is well-formed and hence its resultant γ' is an element of R_w and so true in w. It follows, in virtue of theorem IX, that β and β' have the same value in w'. Thus $V'(\beta,w) = T$. So, for all interpretations V′ which give to all variables other than x the same value as V, $V'(\beta,w) = T$, and so $V(\forall x\beta,w) = T$.

If α, i.e. $\forall x\beta$, does not appear in R_w there is an interpretation V′ which gives to all variables other than x the same values as V, such that β is false in w'. For, if $\forall x\beta$ does not appear in R_w, $\sim\forall x\beta$ appears in R_w and as $\sim\forall x\beta \vdash \exists x\sim\beta$ is derivable, $\exists x\sim\beta$ appears in R_w. But R_w contains a wff of the form $\exists x\sim\beta \supset \sim\beta[y/x]$ where $\sim\beta[y/x]$ is $\lambda x(\sim\beta)y$, this paraformula being well-formed. It follows that $\sim\beta[y/x]$ appears in R_w since $\exists x\sim\beta, \exists x\sim\beta \supset \sim\beta[y/x] \vdash \sim\beta[y/x]$ is derivable. So $\sim\beta[y/x]$ is true in w and $\beta[y/x]$ is false in w. Let V′ be the interpretation which gives x the same value as V gives to y and to all variables other than x the same value as V. We have that β has the same value in w' as $\beta[y/x]$ has in w. $V'(\beta,w) = F$. It follows that $V(\forall x\beta,w) = F$.

If α has the form $\exists x\beta$ and if α occurs in R_w, then there is an interpretation V′ which gives all variables other than x the same value as V, and β is true in w according to V′. (We leave the proof to the reader who can adapt the proof given above for the case where α has the form $\exists x\beta$ and does not appear in R_w.) It follows that $\exists x\beta$ is true in w.

If α, i.e. $\exists x\beta$, does not appear in R_w, then for every interpretation V′ which gives to all variables other than x the same values as V, such that β is false in w according to V′. It follows that $\exists x\beta$ is false in w. (We leave the proof to the reader who can use the proof given above for the case where α has the form $\forall x\beta$ and appears in R_w.)

If α has the form $L\beta$ and if α appears in R_w, then, since $L\beta \vdash \beta$ is derivable β is in R_w and so β is true in w.

Further, in virtue of lemma VII, for any world w', $L\beta$ appears in $R_{w'}$. It follows that for every world w', β is true in w', and from this that $L\beta$ is true in w.

If α, i.e. $L\beta$, does not appear in R_w, $\sim L\beta$ appears in R_w, and as $\sim L\beta \vdash M\sim\beta$ is derivable $M\sim\beta$ appears in R_w. Further for every w', $M\sim\beta$ appears in $R_{w'}$. Suppose that the wff $M\sim\beta$ is the wff $M\sim\beta_n$ (v. paragraph 25) where w' corresponds with R_n, so that $R_{w'} = R_n$. Then since $M(\alpha^* \wedge (M\sim\beta \supset \sim\beta)), M\sim\beta \vdash \sim\beta$ is derivable, $\sim\beta$ is an element of $R_{w'}$. It follows that since w' is the world corresponding to $R_{w'}$, β is false in w' and hence β is false in w' and hence $L\beta$ is false in w.

If α has the form $M\beta$ and if α appears in R_w there is a world w' such that β is true in w'. (We leave the proof to the reader, who can adapt the proof given above for the case where α has the form $L\beta$ and does not appear in R_w.) It follows that $M\beta$ is true in w.

If α, i.e. $M\beta$, does not appear in R_w, then $\sim M\beta$ appears in R_w, and as $\sim M\beta \vdash L\sim\beta$ is derivable $L\sim\beta$

will be in every $R_{w'}$, and from this, for every world w', β is false in w', It follows that $M\beta$ is false in w.

Remark: The proof cannot strictly be said to be by induction on the construction of α, but by induction on wff with an identical structure. Two wff are said to have the same structure if each can be obtained from the other by substitution of free or bound variables. Then, where α has the form $\forall x\beta$ and α is in R_m we can assume that the lemma has been proved, not only for β, but also for $\lambda x(\beta)y$. Note also that, for instance, where α has the form $\forall x\beta$ and is not in R_w we can suppose that the lemma has been proved, not only for $\beta[y/x]$ ($\beta[y/x] = \lambda x(\beta)y$) but also for $\sim\beta[y/x]$. This is clearly legitimate because we have already proved that if the lemma holds for $\beta[y/x]$ it holds for $\sim\beta[y/x]$.

32. *Lemma IX*: The wff α* is quasi-satisfiable in the quasi-universe ⟨D,W,Q⟩.

Proof: Suppose that $M\alpha^*$ is the wff $M\beta_n$ (see paragraph 25.) Then $M\alpha^* \wedge (M\alpha^* \supset \alpha^*)$ is the wff $\gamma_{n.0}$, and it is in R_n. Now, since we have $M\alpha^* \wedge (M\alpha^* \supset \alpha^*) \vdash \alpha^*$ it follows that α* is in R_n and thus is true in w, where w is the world corresponding to R_n.

33. *Lemma X*: The quasi-universe ⟨D,W,Q⟩ is regular.

Proof: For any number m all theorems are in R_w. So all theorems are satisfiable in ⟨D,W,Q⟩. By theorems X and XI ⟨D,W,Q⟩ is regular.

With the proof of theorem XIII we have established that if α* is a consistent wff there is a regular quasi-universe ⟨D,W,Q⟩ in which α* is quasi-satisfiable.

V Completeness of S5[1]

34. Recall that we are given the following: (1) the language S5[1], defined in CLM, 23; (2) the semantic definitions of CLM, 3 and 4, which, as observed in CLM, 24, are applicable to the language S5[1]; (3) theorems I, II and IV of CLM, adapted, as has been said in CLM, 26, to the language S5[1]; (4) a semantic (not quasi-semantic) theorem analogous to theorem IX of the present article; for S5[1] the variables x and w of this theorem are individual variables; (5) the deductive system S5[1] defined in CLM, 27.

35. We can make sets and series of wff of the language S5[1] analogous to the sets and series defined in paragraphs 25-29 of the present article. Lemmas I-VIII can be proved as in those paragraphs.

36. We assume a universe ⟨D,W⟩, and establish the correspondences described in paragraph 30 of the present article. We no longer need the quasi-universe ⟨D,W,Q⟩ containing just those intensional relations which correspond with predicate variables. Lemma VIII can be read as follows:

Lemma VIII: Let ⟨D,W⟩ be a universe and V an interpretation relative to ⟨D,W⟩. Let w be a world in ⟨D,W⟩ corresponding to the set R_w. Let α be a wff. Then V(α,w) = T or F according as α is or is not in R_w.

The proof is as in paragraph 31. We have the truth or falsity of α rather than quasi-truth or falsity because we don't have second-order quantifiers in the language S5[1]. So if one looks at the series of quasi-semantical definitions of truth and falsity given in paragraph 3 of the present article, one can see that they are equivalent to the notions of truth or falsity with respect to a universe ⟨D,W⟩ except for being restricted by Q. In other words, for the language $S5^2$, the definitions of truth and falsity in quasi-universe ⟨D,W,Q⟩ based on a universe ⟨D,W⟩ are no different from those for truth and falsity in ⟨D,W,Q⟩ except where *x* is a propositional or predicate variable. The difference in those cases arises because one doesn't consider all the intensional relations in ⟨D,W⟩, but only those which occur in the quasi-universe ⟨D,W,Q⟩ based on ⟨D,W⟩. It follows that for the language $S5^2$ a wff containing second-order quantifiers might be true or false in *w* without being true or false in *w* in ⟨D,W⟩ with respect to the corresponding V, and vice versa. The absence of second-order quantifiers in $S5^1$, makes this difference disappear.

It follows that we can proceed as follows: Apply the quasi-semantical definitions of paragraph 3 (not those of paragraph 4) to the language $S5^1$. In Lemma IV choose the quasi-universe ⟨D,W,Q⟩, and not the Universe ⟨D,W⟩. Prove, as in paragraph 31, that α is or is not true or false in *w* with respect to an interpretation V according as α is or is not in R_w. We can claim that α is true or false for according as α is or is not in R_w, which is essentially lemma VIII relative to $S5^1$, as formulated above.

37. We can prove, as in paragraph 32:

Lemma IX: α* is satisfiable in a Universe ⟨D,W⟩.

Lemma X falls out of the collection of lemmas I-IX proved for $S5^1$

Theorem XV: If α* is a consistent wff in $S5^1$ then α* is satisfiable.

From this one can conclude

Theorem XVI: $S5^1$ is complete

38. It has been possible to adapt the Henkin proof method to $S5^2$ and $S5^1$. One might have considered adapting the Gödel proof method to $S5^1$.[37] But one encounters a difficulty from the fact that the Gödel method rests on the technique of prenex formulae, and this technique is unavailable in modal logic.[38]

6. Commentary on QA

The completeness proof in QA is a proof of what Bayart calls 'quasi-completeness', defined in terms of 'quasi-universes'. This is because second-order logic is known to be unaxiomatisable when the n-place predicates range over all sets of n-tuples from the domain of individuals.[39] Henkin 1950, however, establishes a form of completeness whereby the range of the n-place predicates can be arbitrarily restricted by a subset of 'allowable' sets of n-tuples. Bayart refers to this set as P_n and calls such a restricted universe a 'quasi-universe'. Following Henkin 1950 (p. 81, note 5), Bayart notes that soundness requires that the quasi-universe be 'regular' in the sense that the range of an n-place predicate must include every n-place condition definable in the language. I.e., we have to have, for every wff α, the validity of:

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[37] The reference here is to Gödel 1930.

[38] The article ends at this point without a bibliography or an indication of an institutional affiliation. (On this see the introduction.)

[39] Henkin 1950, p. 81 points out that this follows from Gödel 1931.

$$\exists\varphi\forall x_1...\forall x_n(\varphi x_1...x_n \equiv \alpha)$$

where φ does not occur free in α. This is necessary to ensure the quasi-validity of principles like ∀I and I∃, as illustrated in the discussion of (2) in the commentary on CLM above.

The completeness proof in Bayart 1959 is a standard Henkin proof, though Bayart only proves what is sometimes known as 'weak completeness'—that every consistent wff has a model rather than that every consistent set of wff has a model. I.e. he proves that where α* is a consistent wff then α* is satisfiable. I shall consider later how his proof might be modified to establish strong completeness. In the modal case satisfiability of course means the existence of a model in which α* is true in some world, for some assignment of values to its free variables. The proof can be divided into three stages. The first stage (QA, 25-27) consists in the construction of a (consistent) set H of wff which has the following properties:

1. H contains $M\alpha^*$
2. For every wff of the form $M\beta$, and for every collection of wff of the form $\exists x_1\delta_1,..., \exists x_n\delta_n$, there are variables $y_1,..., y_n$ such that $M(M\alpha^* \wedge (M\beta \supset \beta) \wedge (\exists x_1\delta_1 \supset \delta_1[y_1/x_1]) \wedge ... \wedge (\exists x_n\delta_n \supset \delta_n[y_n/x_n]))$ is in H. (Note that 'variables' includes individual variables, propositional variables and predicate variables.)
3. Where γ is a fully modalised wff either γ or ~γ is in H.

Before I discuss how Bayart proves that there is such an H, I will look at the second stage of his proof. What has to be achieved in this. In a model for modal predicate logic constructed in accordance with the Henkin method, the 'worlds' correspond to sets of wff in such a way that truth in a world is equivalent to membership of the corresponding set. Because Bayart is concerned only with S5 we don't need to talk about one world's being accessible from another. But we do need to prove that whenever $M\eta$ is in a (set corresponding to a) world w then η itself is true in some world w' in the model, and vice versa. The purpose of H is to provide a recipe for constructing such a set of worlds, and this procedure is described in QA, 28 and 29. Bayart first constructs a family of sets of wff Q_0, Q_1, Q_2, etc. For every wff Mη in H there is one of these Q_ns which contains, not only $M\alpha^*$, but also $M\eta \supset \eta$. That particular Q_n, like every member of Q, also contains, for every $\exists x\delta$, $\exists x\delta \supset \delta[y/x]$ with respect to some variable y of the same type (individual, propositional or n-place predicate) as x is. The need to have wff like $\exists x\delta \supset \delta[y/x]$ is well-known in first-order logic, where it is sometimes referred to as the witness property. Finally each Q_n is extended to a maximal consistent R_n, and it is the R_ns which correspond to the worlds. Bayart puts a set of individuals into 1-1 correspondence with the individual variables, and a set of worlds into 1-1 correspondence with the R_ns. On this basis, in the third stage of his proof, Bayart associates propositional and predicate variables with appropriate intensions, and thus establishes a value assignment V. He then proves by the usual kind of induction that $V(\alpha,w) = T$ iff $\alpha \in R_w$.

I now proceed to indicate how Bayart proves the existence of a suitable H, and then how he constructs the family Q out of it. First assume two separate enumerations of wff. First an enumeration β_0, β_1, ... etc. of all wff. Second a separate enumeration $\exists x_1\delta_1$, $\exists x_2\delta_2$, $\exists x_3\delta_3$, ... etc. of all wff beginning with ∃. Then assume an enumeration of all pairs of natural numbers nm in such a way that if m < k then nm precedes nk, and if n < k then nm precedes km.[40] Associate with each nm a wff γ_{nm}, where the γ_{nm}s look like this:

[40] The regular diagonal representation of pairs of numbers will achieve this.

γ_{00}: $M\alpha^* \wedge (M\beta_0 \supset \beta_0)$	γ_{01}: $\exists x_1\delta_1 \supset \delta_1[y_{01}/x_1]$	γ_{02}: $\exists x_2\delta_2 \supset \delta_2[y_{02}/x_2]$ ...	
γ_{10}: $M\alpha^* \wedge (M\beta_1 \supset \beta_1)$	γ_{11}: $\exists x_1\delta_1 \supset \delta_1[y_{11}/x_1]$	γ_{12}: $\exists x_2\delta_2 \supset \delta_2[y_{12}/x_2]$ ...	
γ_{20}: $M\alpha^* \wedge (M\beta_2 \supset \beta_2)$	γ_{21}: $\exists x_1\delta_1 \supset \delta_1[y_{21}/x_1]$	γ_{22}: $\exists x_2\delta_2 \supset \delta_2[y_{22}/x_2]$ ...	
...	...	...	...

and so on. I.e., the γ_{no}s are the leftmost column, and γ_{nm} is the m'th wff in the nth row. The reason for this series of wff is that for every $M\beta$ we need a world where β is true, and where every true existential wff can be 'instantiated' by a witnessing variable. In this construction the variable y_{nm} is chosen to be one which does not occur in δ_m or in any γ_{hk} where hk is earlier than nm in the enumeration of pairs of numbers.

Let G consist of all wff ζ_{nm}, where $\zeta_{nm} = M(\gamma_{n0} \wedge ... \wedge \gamma_{nm})$. The point of this is to ensure that any finite subset of wff in the nth row is jointly possible. Bayart's proof that G is consistent is by induction on the 'rank' of ζ_{nm}, i.e., by the place of its highest variable in the enumeration of the nm pairs. First prove that $\{\gamma_{00}\}$ is consistent, and then, for the induction, let G_{nm} denote the set of all ζ_{hk} of lower rank than ζ_{nm}, assume that G_{nm} is consistent, and prove that in that case so is $G_{nm} \cup \{\zeta_{nm}\}$.

ζ_{00} is $M(M\alpha^* \wedge (M\beta_0 \supset \beta_0))$

By standard principles of S5, if ζ_{00} is not consistent then

$\vdash M(M\beta_0 \supset \beta_0) \supset {\sim}M\alpha^*$

But since $\vdash M(M\beta_0 \supset \beta_0)$ we have $\vdash {\sim}M\alpha^*$, contradicting the assumed consistency of α^*. So G_{00} is consistent.

Now assume that G_{nm} is consistent but that $G_{nm} \cup \{\zeta_{nm}\}$ is not. There are two cases according as m = 0 or m ≠ 0. If m = 0 then we would have, for some finite subset Λ of G_{nm}, that

$M(M\alpha^* \wedge (M\beta_n \supset \beta_n)),\Lambda \vdash \perp$

By standard principles of S5, bearing in mind that Λ is fully modalised, we would have

$M\alpha^*,\Lambda \vdash {\sim}M(M\beta_n \supset \beta_n)$

But $\vdash M(M\beta_n \supset \beta_n)$ and so

$M\alpha^*,\Lambda \vdash \perp$.

But $\Lambda \subseteq G_{nm}$ and G_{nm} is consistent and contains $M(M\alpha^* \wedge (M\beta_0 \supset \beta_0))$, and so, since $M(M\alpha^* \wedge (M\beta_0 \supset \beta_0)) \vdash M\alpha^*$ this would make G_{nm} inconsistent, which contradicts the induction hypothesis that Λ is consistent, since all its members are of lower rank that γ_{nm}. While I have appealed in this commentary to 'standard principles of S5', Bayart provides all the necessary proofs, as should be apparent from the translation.

For m ≠ 0 we have that ζ_{nm} is

$$M(\gamma_{n0} \wedge ... \wedge \gamma_{nm-1} \wedge (\exists x_m \delta_m \supset \delta_m[y_{nm}/x_m])).$$

Suppose that

$$M(\gamma_{n0} \wedge ... \wedge \gamma_{nm-1} \wedge (\exists x_m \delta_m \supset \delta_m[y_{nm}/x_m])),\Lambda \vdash \bot$$

where Λ is a subset of G_{nm} (all of whose wff are therefore of lower rank that ζ_{nm}). Then

(12) $\gamma_{n0} \wedge ... \wedge \gamma_{nm-1},\Lambda \vdash \sim(\exists x_m \delta_m \supset \delta_m[y_{nm}/x_m])$

And since y_{nm} is not free in Λ or in δ_m or in $\gamma_{n0}, ..., \gamma_{nm-1}$ we have

$$\gamma_{n0} \wedge ... \wedge \gamma_{nm-1},\Lambda \vdash \sim\exists y_{nm}(\exists x_m \delta_m \supset \delta_m[y_{nm}/x_m])$$

But

$$\vdash \exists y_{nm}(\exists x_m \delta_m \supset \delta_m[y_{nm}/x_m])$$

and so

$$\gamma_{n0} \wedge ... \wedge \gamma_{nm-1},\Lambda \vdash \bot.$$

Since Λ is fully modalised we have

(13) $M(\gamma_{n0} \wedge ... \wedge \gamma_{nm-1}),\Lambda \vdash \bot$[41]

i.e. that $\{\zeta_{nm-1}\} \cup \Lambda$ is inconsistent. But all its members are of lower rank than ζ_{nm}, and so are in G_{nm}, which is assumed consistent. So $G_{nm} \cup \{\zeta_{nm}\}$ is consistent. Since G is the union of all the G_{nm} its consistency follows from their consistency in the usual way. Finally let G be extended to H by ordering all modalised wff of $\mathfrak{L}$, and adding each if it is consistent to do so, and its negation if not. This ensures that H has the three properties mentioned above.

H itself does not correspond with any of the worlds — indeed it is not maximal, and all its members are fully modalised. But it can be used to obtain sets which do so correspond. For each n Bayart forms a set Q_n which consists of H together with all the γ_{nm}s for $0 \leq$ m. He then proves that each Q_n is consistent. Suppose it were not. Then there will be some finite subset Λ of Q_n such that $\Lambda \vdash \bot$. Now, among the γs in Λ there will be one, say γ_{nm}, such that no other γ in Λ has a higher rank than γ_{nm}. And in that case, every γ in Λ will appear as a conjunct in $M(\gamma_{00} \wedge ... \wedge \gamma_{mn})$ in H and so $\{\gamma_{00},..., \gamma_{nm}\}$ is consistent, and since $\Lambda \subseteq \{\gamma_{00},..., \gamma_{nm}\}$ then Λ is also consistent.

[41] The model that Bayart is constructing has a constant domain for all worlds and thus validates the Barcan formula BF: $M\exists x\alpha \supset \exists xM\alpha$. Since he is working within S5 BF is provable, but it is interesting to note where this fact is used in his proof. In fact it is at (13), since the proof here claims that since Λ is fully modalised we may move from a sequent of the form $\alpha,\Lambda \vdash \bot$ to one of the form $M\alpha,\Lambda \vdash \bot$. But take α to be $\exists x\varphi x$ and Λ to be $\sim\exists xM\varphi x$. Clearly $\exists x\varphi x,\sim\exists xM\varphi x \vdash \bot$, and so, since $\sim\exists xM\varphi x$ is fully modalised, we obtain, by the principle used to obtain (13), $M\exists x\varphi x,\sim\exists xM\varphi x \vdash \bot$. (In fact it is straightforward to derive BF in Bayart's axiomatisation of S5, since it is straightforward to obtain $\exists x\alpha \vdash \exists xM\alpha$, and since the consequent is modalised we may obtain $M\exists x\alpha \vdash \exists xM\alpha$ by MI.) The structure of the proof given in the text from (12) to (13) is to remove a modal operator using IM, apply certain quantificational principles to the result, and then put the operator back at (13).

Finally each Q_n is extended to a maximal consistent R_n, and it is these R_ns which correspond to the worlds. We note some features of each R_n. First, all the R_ns have the same modalised wff. This is proved in lemma VII. This means that if $L\eta \in R_n$ then it appears in every R_m and so η appears there also. Second, if $M\eta \in R_m$ then, where $M\eta$ is the nth wff beginning with M. $M\eta \supset \eta$, being a conjunct of γ_{n0} will appear in R_n and since $M\eta$, being modalised, will also be in R_n, then η will appear in R_n. So, where $M\eta \in R_m$ then η will appear in R_n. Third, each R_m contains Q_m, and therefore, for every n, where $\exists x\delta$ is the nth wff beginning with $\exists$ there will be some y such that $\exists x\delta \supset \delta[y/x] \in Q_m$. Finally, since $M\alpha^* \in H$, there will be some R_n such that $\alpha^* \in R_n$. All these properties ensure that the model that Bayart constructs in QA, 30 enables the 'truth lemma' he proves there to be established by a standard induction on the construction of wff of $\mathcal{L}$.

In QA, 30, Bayart asks us to assume that we have put the individual variables of $\mathcal{L}$ into a 1-1 correspondence with a denumerable set of individuals.[42] It is more common nowadays to let the domain simply *be* the individual variables, but of course any denumerable domain will do. Bayart perhaps has in mind that while there may be some particular intended domain of individuals, it is not the business of logic to commit to it. He also assumes a 1-1 correspondence between a denumerable set of 'worlds' and the maximal consistent sets R_w. Notice that, in contrast to the worlds in the usual kind of canonical model, there are only denumerably many maximal consistent sets in Bayart's model.

Based on these correspondences Bayart is able to define the P_ns which, together with D and W, constitute the quasi-universe $\langle D,W,Q\rangle$. The intensional relations of each P_n (where propositions are 0-place intensional relations) are those which correspond to predicate variables in the sense that an n-place intensional relation ω is in P_n if there is an n-place predicate variable φ such that, for any $w \in W$, and any individuals $a_1,..., a_n \in D$ which correspond with individual variables $x_1,...,x_n$ of $\mathcal{L}$, $\omega(a_1,...,a_n,w) = T$ iff $\varphi x_1...x_n \in R_w$. This system of correspondences automatically generates a 'canonical' interpretation, which I will here call V*. For V* we have

$$V^*(\varphi x_1...x_n,w) = 1 \text{ iff } \varphi x_1...x_n \in R_w$$

This is because, where $a_1,..., a_n$ correspond to $x_1,..., x_n$ and w corresponds with R_w, by the correspondences assumed at the beginning of QA, 30, $V^*(\varphi)$ is the function ω such that $\omega(a_1,...,a_n,w= T$ iff $\varphi(x_1,...,x_n) \in R_w$. The task of QA, 31 is to prove the 'truth lemma' (Lemma VIII) that for any wff α of $\mathcal{L}$

$$V^*(\alpha,w) = T \text{ iff } \alpha \in R_w.$$

Bayart does not signal the canonical interpretation in any way, and does not reserve a special name for it. He does say at QA, 30 that the quasi-universe permits the establishing of other interpretations, and in fact reference to these is necessary at the induction step for the quantifiers in the proof of Lemma VIII. Here is why. If $\forall x\beta$ is in R_w then we have to prove that $V(\beta,w) = T$ for every V just like V* except for what it assigns to x, where x is any kind of variable. So $V(x)$ is either an individual from D, or else an n-place intensional relation in P_n, for $n \geq 0$. Then there will be a variable y which corresponds with $V(x)$, so that $V(x) = V^*(y)$. By $\forall$I we have that $\beta[y/x]$ is in R_w. We have of course to make provision for first forming a bound alphabetic variant of β which contains no y quantifier which would prevent y from occurring free when it replaces x in β. Bayart presents $\alpha[y,x]$ as a paraformula, $\lambda x(\beta)y$, and uses his notion of a 'well-

[42] Bayart's word is «Êtablissons», 'Let us establish', by which he indicates that it is up to us to decide just which correspondence to use.

formed' paraformula as defined at CLM, 7 to guard against the possibility of the accidental binding of *y*. Notice that since *y* is a variable the resultant of $\lambda x(\beta)y$ is just $\beta[y/x]$. Since $\beta[y/x]$ is in R_w then $V^*(\beta[y/x],w) = T$, and then Bayart uses the theorems of QA, II to establish that $V(\beta,w) = T$, and so that $V^*(\forall x\beta,w) = T$. The converse is more straightforward, and requires no additional commentary.

It should be observed that in the *completeness* proof, and in particular in the proof of the truth lemma, no mention is made of the need for the quasi-universe to be regular. It is however required for soundness, but this easily follows from the truth lemma, given the derivability of the comprehension principle, that for any wff α not containing free φ

$$\exists\varphi\forall x_1...\forall x_n(\varphi x_1...x_n \equiv \alpha)$$

is derivable, and so is in every R_w, and so is valid. This fact is stated in Theorem X in QA, 33.

Now to the matter of strong completeness. We know of course that strong completeness (or strong quasi-completeness) holds for modal predicate S5, but the question of interest here is how much adaptation Bayart's proof needs to accommodate it. For strong completeness, in place of a consistent wff α^* assume a consistent set A* of wff. In this extension we need to allow the antecedent and consequent of a sequent to include infinitely many wff. Let A* be a set in a language $\mathfrak{L}_0$ of modal predicate logic, and let $\mathfrak{L}$ be a language with infinitely many new variables (of all types) not in $\mathfrak{L}_0$. We then assume that the ordering of variables used in Bayart's proof only concerns the variables of $\mathfrak{L}$ which are not in $\mathfrak{L}_0$. The principal idea behind what follows is this. Where Bayart uses the single wff $M\alpha^*$ as a component of each of the wff ζ_{nm} which make up G we now have to use a whole family Γ_{nm} of wff, where each member of Γ_{nm} has as a component $M\alpha$ for some conjunction α of wff in A*. Just as the ζ_{nm}s can be enumerated on the basis of the enumeration of the nm pairs, so can the Γ_{nm}s. We then make G the union of all these Γ_{nm}s. Specifically we proceed as follows. Change the definition of each γ_{n0} so that it is simply $M\beta_n \supset \beta_n$. Now assume that α is a conjunction of wff from A*, and let $\zeta_{nm}[\alpha]$ be $M(M\alpha \wedge \gamma_{n0} \wedge ... \wedge \gamma_{nm})$. Let Γ_{nm} be the set of all $\zeta_{nm}[\alpha]$, where α is any conjunction of wff in A*, and let G_{nm} be the set of all wff $\zeta_{hk}[\alpha]$ (for every conjunction α of wff in A*) whose rank is lower than nm. From this definition it follows that, where γ_{nm} is $\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m]$, then y_{nm} does not occur in δ_m or in any member of G_{nm}.

We prove that each G_{nm} is consistent. The proof is by induction on the rank of Γ_{nm}. G_{00} will be the set of all $M\alpha$ where α is a conjunction of wff in A*. G_{00} is consistent, since A* is consistent. Now suppose that m = 0 and n > 0. Since G_{n0} is of lower rank than Γ_{n0} we assume for induction that G_{n0} is consistent. Suppose that $G_{n0} \cup \Gamma_{n0}$ is inconsistent. Then, for some α_1, ..., α_k such that each α_i $(1 \leq i < k)$ is a conjunction of members of A*, and for some $\Lambda \subseteq G_{n0}$, you would have

$$M(M\alpha_1 \wedge (M\beta_n \supset \beta_n)), ..., M(M\alpha_k \wedge (M\beta_n \supset \beta_n))),\Lambda \vdash \bot.$$

So, since Λ is fully modalised, by principles of S5:

$$M\alpha_1,..., M\alpha_k,\Lambda \vdash \bot.$$

But this would make G_{n0} inconsistent, since $\Lambda \subseteq G_{n0}$ and $M(\alpha_1 \wedge ... \wedge \alpha_k)$ is in G_{00} and $G_{00} \subseteq G_{n0}$.

For m ≠ 0, suppose that G_{nm} is consistent, but that $G_{nm} \cup \Gamma_{nm}$ is not. That would mean that there will be some $\Lambda \subseteq G_{nm}$, and some wff α_1, ..., α_k which are conjunctions of wff from A*, such that, where γ is $(\gamma_{n0} \wedge ... \wedge \gamma_{nm-1})$,

$$M(M\alpha_1 \wedge \gamma \wedge (\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m])),..., M(M\alpha_k \wedge \gamma \wedge (\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m])),\Lambda \vdash \bot.$$

So

$$M\alpha_1 \wedge ... \wedge M\alpha_k \wedge \gamma,\Lambda \vdash \sim(\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m]).$$

So, since y_{nm} does not occur in δ or in any wff in G_{nm}, we have

$$M\alpha_1 \wedge ... \wedge M\alpha_k \wedge \gamma,\Lambda \vdash \sim\exists y_{nm}(\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m])$$

But $\vdash \exists y_{nm}(\exists x_m\delta_m \supset \delta_m[y_{nm}/x_m])$, and so

$$M\alpha_1 \wedge ... \wedge M\alpha_k \wedge \gamma,\Lambda \vdash \bot$$

and so, since all members of Λ are modalised,

$$M(M\alpha_1 \wedge ... \wedge M\alpha_k \wedge \gamma),\Lambda \vdash \bot$$

But this would make G_{nm} inconsistent, since $M(M(\alpha_1 \wedge ... \wedge \alpha_k) \wedge \gamma)$ is in G_{nm} and $\Lambda \subseteq G_{nm}$.

Let G be the union of all the G_{nm}s. Since each G_{nm} is consistent then so is G.

In order to ensure that there is a world satisfying A* we add to the Q_ns an extra set Q_{A*}, which includes every member of A* together with every γ_{1m} for m ≠ 0. (The choice of 1 is arbitrary here, and is only for definiteness.) We shew that Q_{A*} is consistent. Suppose it were not. Then you would have $\Lambda \vdash \bot$ for some finite subset Λ of Q_{A*}, which contains some $\alpha_1,..., \alpha_k$ from A*, and some finite collection of the wff γ_{1h}, for h ≠ 0. Among these there will be a greatest, say γ_{1m}. Now suppose that $M(\alpha_1 \wedge ... \wedge \alpha_k)$ is β_n. Then let Λ′ be

$$\{M\beta_n, (M\beta_n \supset \beta_n), \gamma_{11}[y_{n1}/y_{11}], ..., \gamma_{1m}[y_{nm}/y_{1m}]\}$$

Clearly if $\Lambda \vdash \bot$ then $\Lambda' \vdash \bot$ since none of the y_{ni}s occur in Λ. But

$$M(M\beta_n \wedge (M\beta_n \supset \beta_n) \wedge \gamma_{11}[y_{n1}/y_{11}] \wedge ... \wedge \gamma_{1m}[y_{nm}/y_{1m}])$$

is in G_{nm} and therefore in H. So Λ is consistent.

I have tried, in this extension of Bayart's proof to the case of strong completeness, to use methods which seem no more elaborate than those found in Bayart's own proof. This should give at least an indication of how Bayart's proof might be adapted to the case of strong completeness, even though Bayart himself does not consider doing so.

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