

# Sur les Cas Stratifiés du Schéma de Remplacement

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A note (\*) by Jean Coret, presented by Mr. Jean Leray  
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We will establish the following result: every stratified instance of the replacement scheme is provable in the set theory of Zermelo ( $Z^-$ ) with neither the axiom of choice nor foundation.

Upper-case Roman letters  $F, G, \dots$  will denote operators definable in  $Z^-$ ;  $F \circ G$  is the composition of  $F$  and  $G$ ;  $F^{(k)}$  is the operator  $F$  iterated  $k$  times, with  $F^{(0)} = \mathcal{I}$ , the identity. If  $F$  is a permutation of the universe,  $F^{-1}$  is the inverse of  $F$ .

**DEFINITION 1**  $F$  is permitted if

$$(\forall a)(\exists b)(\forall y)(y \in b \longleftrightarrow (\exists x \in a)(y = F(x)))$$

If  $F$  is permitted, let  $F^*$  be the operator

$$a \mapsto \{Fa : x \in a\}$$

For example, in  $Z^-$ , the operators  $\bigcup$  (sumset) and  $\mathcal{P}$  (power set) and  $\{\cdot\}$  (singleton) are permitted. If  $F$  and  $G$  are permitted so is  $F \circ G$ . If  $F$  is permitted, so is  $F^*$ , and we will write as  $F^{[k]}$  for the suite of operations defined by

$$F^{[0]} =: \mathcal{I}; \quad F^{[1]} =: F; \quad F^{[k+1]} =: (F^{[k]})^*.$$

**DEFINITION 2**  $F$  is an admissible permutation if both  $F$  and  $F^{-1}$  are permitted.

We can now define the suite  $F_k$  by:

$$F_0 =: \mathcal{I}; \quad F_1 = F; \quad F_{k+1} =: (F_k)^* \circ F_1.$$

By induction on  $k$  one proves immediately: if  $F$  is an admissible permutation, then the  $F_k$  are all admissible permutations.

See note 1.

**LEMMA 3** *If  $F$  is an admissible permutation,  $\Phi$  a stratified formula with free variables  $x_1 \dots x_n$ ,  $\sigma = (\sigma_1 \dots \sigma_n)$  be a suite of integers stratifying  $\Phi$ ,  $\Phi'$  the result of replacing  $x_i \in x_j$  by  $x_i \in F(x_j)$  throughout, then*

$$(\forall x_1 \dots x_n)[\Phi(x_1 \dots x_n) \longleftrightarrow \Phi'((F_{\sigma_1})^{-1}x_1 \dots (F_{\sigma_n})^{-1}x_n)]$$

*Proof:*

We prove lemma 1 by induction on the complexity of  $\Phi$ . ■

We can infer from it results analogous to those in reference [1].

**LEMMA 4** *Let  $F$  be an admissible permutation. Then the following hold for every integer  $k$ .*

1. *If  $F$  is the identity on  $\bigcup^k x$ , then  $F^{[k+1]}x = x$ ;*
2.  *$\bigcup^k \circ F^{[k+1]} = F^* \circ \bigcup^k$*

**LEMMA 5** *With the notations of lemma 4, if  $\Phi$  is stratified then for all integers  $m$*

$$(\forall x_1 \dots x_n)[\Phi(x_1 \dots x_n) \longleftrightarrow \Phi((F^{[\sigma_1+m]}x_1 \dots (F^{[\sigma_n+m]}x_n)))]$$

*Proof:*

We prove lemma 5 by noting that if  $\sigma$  stratifies  $\Phi$ , so does the sequence  $\sigma + m$ . ■

**DEFINITION 6** *Let*

- $\Phi$  be a formula with free variables among  $x_1 \dots x_n$ ;
- $K$  be a subset of  $\{1, 2 \dots n\}$ ;
- $n_i$  be the number of occurrences of  $x_i$  in  $\Phi$ ;
- $y_{j,k}$ , for  $j = 1, \dots, n_k$  and  $k \notin K$  a suite of variables not occurring in  $\Phi$ .

*Then  $\Phi_K$  is the expression obtained from  $\Phi$  by replacing the  $j$ th occurrence of  $x_k$ —for  $k \notin K$ —by  $y_{j,k}$ . If  $\Phi_K$  is stratified we will say that  $\Phi$  is stratified on the set  $K$ , or—further—that the variables  $x_i$  with  $i \in K$  are stratified in  $K$ . If  $K$  is empty we will say that  $\Phi$  is pseudo-stratified.*

See note 5.

For example:

$$\Phi(x_1) : (\forall t)(\forall t')((t \in t' \wedge t' \in x) \rightarrow t \in x)$$

Now for  $K = \emptyset$  we have

$$\Phi_{\emptyset}(y_{1,1}, y_{2,1}) : (\forall t)(\forall t')((t \in t' \wedge t' \in y_{1,1}) \rightarrow t \in y_{2,1})$$

The expression “ $x$  is a transitive set” is accordingly pseudo-stratified.

**COROLLARY 7** *If  $\Phi$  is stratified in  $K$  and  $F$  is an admissible permutation then, for every integer  $m$*

$$\forall x_1 \dots x_n [\Phi(x_1 \dots x_n) \longleftrightarrow \Phi_K(\dots F^{[\sigma_i+m]}x_i \dots F^{[\sigma_j, k+m]}x_k \dots)]$$

for  $i \in K, k \notin K, j \in \{1 \dots n_k\}$  and  $\sigma$  a stratification of  $\Phi_K$ .

See note 2.

**LEMMA 8** *If the variable  $x_{n+1}$  is stratified in  $\Phi$ , and  $\sigma$  stratifies  $\Phi_{\{n+1\}}$  then the following holds:*

if

$$(\Phi(x_1 \dots x_n, x_{n+1}) \wedge \Phi(x_1 \dots x_n, x'_{n+1})) \rightarrow x_{n+1} = x'_{n+1}$$

then  $\Phi(x_1 \dots x_n, x_{n+1}) \rightarrow x_{n+1} \in \mathcal{P}^{\sigma_{n+1}+1}(v_1 \cup \dots \cup v_n)$ ,

where

$$v_i =: \bigcup_{j=1 \dots n_i} \bigcup^{\sigma_{j,i}} x_i; \quad i = 1, \dots, n$$

*Proof:*

For every integer  $q$  we have  $x \subseteq \mathcal{P}^q \circ \bigcup^q x$ , and because  $\mathcal{P}^q$  is  $\subseteq$ -increasing it will suffice to prove  $u \subseteq v$  where

$$u =: \bigcup^{\sigma_{n+1}} x_{n+1} \quad \text{and} \quad v =: v_1 \cup \dots \cup v_n.$$

If  $u \not\subseteq v$  then we can find  $a$  and  $b$  such that  $a \in u, a \notin v$ , and  $b \notin u \cup v$ .

See note 8.

Now let  $F$  be the transposition  $(a, b)$  (that swaps  $a$  with  $b$  while fixing everything else). It is easy to check that  $F$  is admissible.  $F$  is the identity on  $v_i$ , so—by lemma 4 part (1)—we have

$$F^{[\sigma_{j,i}+1]}(x_i) = x_i \tag{1}$$

Further,  $\bigcup^{\sigma_{n+1}} \circ F^{[\sigma_{n+1}+1]}(x_{n+1}) \neq \bigcup^{(\sigma_{n+1})}(x_{n+1}) = u$

On the one hand—by lemma 4 part (2)—we have

$$\bigcup^{(\sigma_{n+1})} \circ F^{[\sigma_{n+1}+1]}(x_{n+1}) = F^* \circ \bigcup^{(\sigma_{n+1})}(x_{n+1}) = F^*(u)$$

On the other hand  $u \neq F^*(u)$  by construction of  $F$ . Therefore

$$F^{[\sigma_{n+1}]}(x_{n+1}) \neq x_{n+1} \tag{2}$$

Now, by corollary 7, we obtain

$$\phi(x_1 \dots x_n, x_{n+1}) \longleftrightarrow \phi_{\{n+1\}}(\dots F^{[\sigma_{j,i}+1]}(x_i) \dots F^{[\sigma_{n+1}+1]}(x_{n+1})).$$

Formula (1) now gives us

$$\phi(x_1 \dots x_n, x_{n+1}) \longleftrightarrow \phi(x_1 \dots x_n, F^{[\sigma_{n+1}+1]}(x_{n+1})).$$

so, by the hypothesis on  $\phi$ ,  $x_{n+1} = F^{[\sigma_{n+1}]}(x_{n+1})$ , giving a contradiction. ■

See note 3.

From lemma 8 we infer

**COROLLARY 9** *In every extension of  $Z^-$  every set definable by a stratified formula without parameters is hereditarily finite.*

**COROLLARY 10** *If the variable  $y$  is stratified in  $\phi$  then*

see note 4

$Z^- \vdash (\forall x)(\forall y)(\forall z)[\Phi(x, y) \wedge \Phi(x, y') \rightarrow y = y'] \rightarrow (\forall a)(\exists b)(\forall y)(y \in b \leftrightarrow (\exists x \in a)\Phi(x, y))$

The axiom of replacement is therefore provable in  $Z^-$ : we use only the axiom of comprehension for  $\Phi$ .

See note 6.

Remark:

It is easy to find examples of pseudostratified formulæ for which replacement is not provable in  $Z^- +$  Axiom of foundation  $+$  Axiom of Choice. For example, the formula with two free variables  $x$  and  $y$  saying that  $x$  is a wellordering and  $y$  an isomorphic ordinal.

See note 9.

## References

[1] Coret, J. *Comptes Rendues Acad. Sc. Paris*, tome **264** série A, 1967, p 837.

## Notes

Although in general I have adhered slavishly to Coret’s notation, there are limits. Where he writes ‘ $U$ ’ I write ‘ $\cup$ ’ since that is obviously what he meant. (Why did the *Comptes Rendues* use a printing house that lacked this symbol?)

He also uses a large boldface ‘**C**’ for complementation, which I have removed by rewriting formulæ so as not to need any notation for complementation.

I have exploited the  $\text{\LaTeX}$  machinery to provide numbers for all his definitions and lemmata, not just those that he numbers.

Coret always writes “integer” (“*entier*”) even in circumstances where one would have expected him to consider only natural numbers. I have translated *entier* as *integer* throughout. I have inserted ‘Proof’ at the start of a proof and ‘■’ at the end of a proof.

## Note 1

In the later literature the word *setlike* is used for functions that have this nice property. Specifically a function  $f$  is 1-setlike if  $f^{\ulcorner}x$  exists for all  $x$ . Coret presumably had in mind the observation that if  $f$  is 1-setlike then so is  $j(f) =: \lambda x.f^{\ulcorner}x$ . This is because  $\{f^{\ulcorner}y : y \in x\}$  is  $\{z \in \mathcal{P}(f^{\ulcorner}\cup x) : (\exists w \in x)(z = f^{\ulcorner}w)\}$  and therefore exists by power set and separation applied to  $f^{\ulcorner}\cup x$ —which exists because  $f$  is 1-setlike.

A function is *setlike* if  $j^n(f)$  is setlike for all  $n$ . What this argument shows is that any *definable* function is setlike. In ZF the axiom scheme of replacement makes it obvious that any definable function is setlike. What is striking about the result here is that it holds for Zermelo set theory as well.

## Note 2

In the antecedent of the conditional of corollary 10 the ‘ $\Phi$ ’ is in lower case. This is presumably a missprint.

## Note 3

The extra information in this corollary and the preceding definition is of no use, at least not here. It may be that this gadgetry is a good way to start thinking about degrees of dysstratification of formulæ but that is not our concern here. The point to notice for the moment is that we can prove lemma 8 by relettering followed by substitution. The meat of the construction is the transposition.

## Note 4

By “ $y$  is stratified in  $\Phi$ ” Coret means that there is a stratification of  $\phi$  which, although it might require distinct occurrences of the other free variables to be given distinct types, nevertheless it treats ‘ $y$ ’ as a bound variable, in that it gives all its occurrences the same type.

## Note 5

Nowadays “pseudostratified” is universally called “weakly stratified”.

## Note 6

This is of course a slip of the pen. He means: the *stratified* instances.

## Note 7

I think nowadays we would say *monotone* rather than *increasing*.

## Note 8

This construction exploits a background assumption that there is no universal set.

## Note 9

A more illuminating observation at this juncture might be that this proof works only for stratified *replacement*; Mathias’ counterexample shows that stratified *collection* is not a theorem of Zermelo. (“For every  $n \in \mathbb{N}$  there is a set of  $n$  infinite sets all of different sizes”).