"On the limits of determinacy in third-order arithmetic and extensions of Kripke-Platek set theory : an introductive study of topics in reverse mathematics, constructibility and determinacy"

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ABSTRACT

This thesis explores the foundations of mathematics, studying determinacy axioms derived from game theory. It exposes their relationship with second-order and third-order arithmetic, examining a significant paper in the field by Montalbán and Shore. The thesis begins by introducing the background concepts of determinacy and second-order arithmetic. It also explains the translation between set theory and second-order arithmetic. Subsequently, the thesis delves into advanced concepts in set theory and subsystems of second-order arithmetic that are crucial for understanding and proving the results proved by Montalbán and Shore. In particular, it presents a generalisation of their second theorem, formulated within a Kripke-Platek set-theoretic version of third-order arithmetic, offering an original contribution to the field. Overall, this thesis contributes to understanding determinacy axioms and their limitations. It provides a synthesis of the foundational knowledge required to comprehend the research conducted by Montalbán and Shore in this specific area.

Faculté des sciences

On the Limits of Determinacy in Third-Order Arithmetic and Extensions of Kripke-Platek Set Theory

An introductive study of topics in reverse mathematics, constructibility and determinacy

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Introduction

How much and which of the axioms at the foundation of mathematics can be stated in a simple game theoretic formalism? To answer this question, one has to investigate areas of mathematical logic such as the foundation of mathematics, reverse mathematics, determinacy of infinite games and many others that overlap with the present thesis. It all traces back to an old story.

At the beginning of the 19th century, the mathematician David Hilbert wanted to answer the disturbing paradox in the theories of the foundation of mathematics from Frege and Russel through his famous program. The aim was to prove once and for all that mathematics, built based on arithmetic and finitary concepts, was complete and consistent. The hopes of accomplishing such a program were destroyed in 1930 and 1931 when Kurt Gödel posted the proofs of his first and second incompleteness theorem, which became famous nowadays. However, even if the enterprise to find universal and unique foundations for all mathematics was shut down, these theorems also opened the door to the analysis of various families of theories in which a certain amount of mathematics can be carried in. Furthermore, the theories T such as the ones that satisfy the hypotheses of Gödel's incompleteness theorems were the kind of ones that every mathematician used to do mathematics with. Thus, it was the beginning of growing interest in the problems indemonstrable from T given by Gödel's theorem.

An example of such a question that was early on treated is: Given a family of sets of reals, do the sets of this family have nice measurability properties? The question was usually asked about projective sets $\undertilde{\Sigma}^1_n$ in the Baire space. That is where the study of determinacy, an axiom about infinite games in ω^ω introduced by Mycielski and Steinhaus began to gain interest when Solovay and Blackwell used it to solve similar questions of descriptive set theory (see [26, 40]) in 1967. It was only a matter of time before other mathematicians like Addison, Martin, Moschovakis and others used the axiom of determinacy as a starting point to demonstrate, for example, the measurability of some class of projective sets. From this point of view, determinacy axioms are very related to the existence of large cardinals, as it is exposed in the book of Kanamori on the subject [25], the historical survey of Larson [31] or the recent paper of Sandra Müller [41]. There are plenty of examples of undecidable questions in ZFC, also in various other fields than descriptive set theory like the continuum problem in set theory, the Whitehead problem [46] in group theory, the Borel conjecture [9] in measure theory, Kaplansky's conjecture on Banach algebras [8], the Brown-Douglas-Fillmore problem [13] on operator algebras, etc.

Another child of this collapse of Hilbert's program which also arises in the 1970s is "reverse mathematics" under the impulse of Friedman [15, 16] and Simpson [48]. This new field of logic, a program in the foundation of mathematics, has as aim to answer the question: "What are the appropriate axioms to prove the theorems of mathematics?". The stronger the axioms, the more difficult it is for a system to meet its requirements. For that reason, the article "Reverse Mathematics: The Playground of Logic" [47] lives up to its name, since reverse mathematics is like a mathematical treasure hunt; trying to find the largest set of systems in which a result remains true. This hunt is typically led within the realm of second-order arithmetic, where everything is natural numbers and sets of natural numbers. This language can express most ordinary, or undergraduate mathematics. On the other hand, the traditional approach to expressing mathematical objects is through set theory, specifically the theory of Zermelo and Fraenkel inside of which the idea of reverse mathematics can be generalised.

In the present thesis, we will try to contribute to answering the questions: What are the appropriate axioms to do mathematics? In order to prove determinacy? What are the appropriate determinacy axioms for expressing part of the theory of second-order arithmetic? etc. More precisely, we will focus on the limit of determinacy in second-order arithmetic and the bound that is exhibited in the paper of Montalbán and Shore [38] who proved two theorems of interest from a reverse mathematics point of view. The first one establishes an upper bound in terms of provability strength in second-order arithmetic when comparing comprehension schemes to determinacy.

Theorem 1. *For each $m \geq 1$, $\Pi^1_{m+2}\text{-}\mathsf{CA}_0 \vdash (\Pi^0_3)_m\text{-}\mathsf{Det}$.*

The second one is a close lower bound in the same terms.

Theorem 2. *For every $m \geq 1$, $\Delta^1_{m+2}\text{-}\mathsf{CA}_0 \nvdash (\Pi^0_3)_m\text{-}\mathsf{Det}$.*

Even if we don't get a natural axiomatic statement equivalent to $(\Pi^0_3)_m$-Det, in the sense of reverse mathematics they give a narrow gap inside of which this weak level of determinacy is located. This way, we have a precise idea of the very limit of determinacy of infinite games that is provable only using natural numbers and sets of natural numbers (Z_2). Along the way, it shows that determinacy is an example of natural theorems provable in Z_2 that requires very strong subsystems of second-order arithmetic, while the huge majority of them known so far can usually be proven by one of the big fives: RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. In the end, we will wonder if the second theorem is generalisable to third-order arithmetic and more.

As Martin highlights in his book about infinitely long games [35] and Montalbán and Shore emphasize at the beginning of their paper, second-order arithmetic can be thought of as ZFC^- (ZFC without power set axiom) or $\mathsf{ZC}^- + \Sigma_1$ REPLACEMENT in a conservative way, concerning a precise translation between statements in second-order arithmetic and set theory. To show their second theorem, Montalbán and Shore used set-theoretic models of $V = L + \mathsf{KP}$, constructible Kripke-Platek theory presented by Barwise in [3], with some amount of SEPARATION and COLLECTION added so that it is a β-model of Δ^1_{m+2}-CA_0. As Hachtman did in [17] to refine the result of Martin according to which $\Sigma^0_{1+\alpha+3}$ requires $\alpha + 1$ iteration of the power set axiom for $\alpha < \omega_1$, this is the starting point of a wilder question in the centre of this thesis: What is the limit of determinacy that we can prove in arithmetic of nth order $(2 \leq n)$, thought as $\mathsf{ZFC}^- + \mathcal{P}^{n-1}(\omega)$ exists? And to what extent are the theorems of Montalbán and Shore generalisable?

Our aim throughout this thesis is to guide the reader from a very elementary level of knowledge from the point of view of mathematical logic (basics of first-order logic, model theory and set theory) to the cutting-edge research in the questions related to the reverse mathematics analysis of determinacy axioms (for some Borel sets). This is indeed the path that was followed by the writer during the redaction of the thesis.

In the first chapter, we will mainly set up all the preliminary notions. We define determinacy and prove some well-known folklore results to familiarize ourselves with it. Then we define second-order arithmetic and the big five, and we present the reverse mathematics results of Steel [49]. Finally, in the last section of chapter 1, we expose the results of the chapter about β-models of [48], showing how one can translate between the formalism of set theory and second-order arithmetic, which is especially useful in proving the theorems of Montalbán and Shore.

In the second chapter, we introduce all the necessary more advanced facts about set theory and subsystems of second-order arithmetic that have to be used in these proofs.

Finally, in chapter 3, we present the proofs themselves, in both cases beginning with a "warm-up" with an easier theorem following the same kind of idea. For the second version, we propose a generalisation stated in the framework of the Kripe-Platek set theory. The latter constitutes an original contribution of the thesis to the field, in addition to a somewhat unique synthesis and introduction to this very specific subject treated by Montalbán and Shore.

Let us end up this introduction with a quotation from Kurt Gödel, that casts a premonitory light on the study of axioms such as the ones of determinacy.

> "There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory."

— Kurt Gödel in "What is Cantor's continuum problem?" (1964), in Kurt Gödel's Collected Works, Vol. II, Feferman, Solomon, Eds.

Contents

Chapter 1

Theories and Languages in which we Play

In order to do mathematics, we need to encode precisely what we are talking about in a formal *language*, by use of constant, function and relation symbols. These are the atoms of any so-called mathematical object. Any universe giving an appropriate interpretation of this language will then be called a *structure* for the language, a place where mathematical objects can live. The outer world where this structure itself lives is a metamathematical question to which everyone is free to give their answer, insofar as the latter allows us to envisage such a structure. We can then use this language along with a dedicated alphabet, logical connectives and quantifiers to state properties and conjectures about these objects.

However, the work of a mathematician is not only to state properties but also to prove them. To do so, like all other scientists, we need to set up a reasonable *theory* that accounts for the behaviour of the objects we intend to model. The basic tenets of such a theory are called its *axioms*. But unlike any other scientist, we will then only use *logical laws* to derive properties of this theory from its axioms, which we call *theorems*.

What characterises the abstract method of mathematics is that any theorem provable from a theory in a particular language using our logical laws will be true in any world, any structure behaving according to the rules of this theory, which we then call *models of the theory*. This foundational and metamathematical fact is the completeness theorem to which the soundness theorem is added. The interested reader can find further details about these foundational concepts in [33] and [12].

It turns out that most mathematics can be unified, at least at first glance, in the language of sets, L_{Set}, which only has the one well-known relation symbol "$\in$" and where all mathematical objects are understood as sets. The very powerful theory within which most mathematicians then work is due to Zermelo and Fraenkel, using the logical rules of classical logic, as we will always do in the course of this study. We now give a reminder of its famous axioms. These are discussed in detail in [23], where the reader may also find the basic development of classical set theory such as ordinal numbers, transfinite induction, etc. In the following, we assume familiarity with the topics discussed in [12, 23, 33] that is, fundamentals of logic, set theory and model theory, for which the previous paragraphs act as a popularizing trailer.

Definition 1.0.1 (ZFC). *The L_{Set}-theory axiomatized by the axioms*

EXTENSIONALITY:	*If X and Y have the same elements, then $X = Y$,*
PAIRING:	*The unordered pair of X and Y, written $\{X,\ Y\}$, is a set,*
UNION:	*The union $\bigcup X$ is a set,*
REGULARITY:	*If X is nonempty, then it has an $\in$-minimal element,*
SEPARATION:	*The collection $\{x \in X \mid \phi(x)\}$, of the elements of X satisfying ϕ is a set,*
REPLACEMENT:	*If $\phi(x, y)$ defines a function on X, then its range, $\{y : (\exists\, x \in X)\ \phi(x, y)\}$, is a set,*
INFINITY:	*There exists an inductive set,*
POWER SET:	*The power set $\mathcal{P}(X)$ is a set,*
CHOICE:	*If no element of X is the empty set, then one of the choice functions of X is a set,*

is called Zermelo-Fraenkel with choice.

Concerning the axiom of infinity, an inductive set is a set S such that

$$\emptyset \in S \quad \text{and} \quad (\forall x \in S)\ x \cup \{x\} \in S.$$

Such a construction will naturally give birth (at least) to the standard set of natural numbers, ω, indubitably an infinite set. We point out that, in the present setting, the axiom of pairing can be deduced from INFINITY, SEPARATION and REPLACEMENT together. Indeed, we can use SEPARATION to extract from S a set with two elements, lets say $2 = \{0, 1\}$ and then apply REPLACEMENT with the formula

$$\phi(x, y) \equiv (x = 0 \wedge y = X) \vee (x = 1 \wedge y = Y)$$

to get the pairing$\{X, Y\}$. Furthermore, this gives us an insight into the essential value of REPLACEMENT in itself, used for such glueing (in an infinite fashion). However, we will keep PAIRING for the sake of uniformity with the remaining of the present study, because in weaker set theories, we won't always take INFINITY for granted. We can reason similarly to show that in some not-too-demanding contexts (as the majority of models of set theory), we can deduce SEPARATION from REPLACEMENT as well. The axiom of choice, in particular, can lead to some astonishing results, such as the famous Banach-Tarski theorem (see [1]). Despite that, it is essential in the proof of many useful algebraic and analytic results. However one can decide to work without it and the theory thus formed will be called ZF. Other variants are, for instance, ZFC^-, where we remove POWER SET, or ZC if we remove REPLACEMENT. Sometimes ZFC^- also has a weaker version of REPLACEMENT that we will see in the next chapter.

The reader can find an extensive development of the classical set theory ZFC in the book of Kuratowski and Mostowski [30].

Remark 1.0.2. When we state an axiom of a theory, we always take implicitly the universal closure for the free variables and formulae of the language of this theory (they may have parameters, encoded by free variables, if not specified otherwise). Taking the universal closure on formulae means the axiom is an infinite set of sentences, called an *axiom scheme*, consisting of all statements of itself, each of them determined by the choice of any suitable formula to state it.

1.1 Playing Infinite Games

Despite the powerful axioms of ZFC, it appears that some natural problems, in particular in descriptive set theory, remain unsolvable, even with strong hypotheses such as the existence of measurable cardinals (see [25, 40]). This led some mathematicians such as Blackwell in 1967 and Addison and Martin after him, to use some unexpected kind of axioms of interest called determinacy axioms. These were first stated by Mycielski and Steinhaus in 1962, though already used before in the 20th century in the proof of theorems aiming to avoid some of the more unpleasant consequences of CHOICE, and will be our main subject of interest for the remainder of this study. The inspiration for it comes from game theory, so let's first define an important tool of this field, namely trees. The results presented in this section can be found in any good book of descriptive set theory like [26] or [40] and the historical context of the arising of determinacy is discussed in [31].

Definition 1.1.1 (Long sequences)**.** *Let B^A symbolise the set of functions $f\colon A \to B$, for any sets A and B. We also write*

$$A^{<\beta} := \bigcup_{\alpha<\beta} A^\alpha$$

for any ordinal β. Finally, $|f|$ will stand for the function's domain (usually, an ordinal).

1. *For any ordinal numbers $\alpha < \beta$, given a function $f : \beta \to A$, we define*

$$f[\alpha] \; : \; \alpha \; \to \; A$$

as the α-length initial segment of f. Conversely, an extension of f is any function in which f is an initial segment.

2. *Given $f : \alpha \to A$ and $\alpha < \beta$, the β-cylinder around f is*

$$[\![f]\!]_\beta \coloneqq \{g \in A^\beta : g[\alpha] = f\},$$

the set of all extensions of f to β. We will omit the β subscript when the context is clear.

3. *For any ordinals β, γ, given $f : \beta \to A$ and $g \colon \gamma \to A$, we define*

$$\begin{aligned} f^\frown g \colon \beta + \gamma \to A : \\ \eta \mapsto \begin{cases} f(\eta) & \textit{if } \eta < \beta, \\ g(\kappa) & \textit{if } \eta = \beta + \kappa \textit{ for some } \kappa, \end{cases} \end{aligned}$$

the concatenation of f and g.

4. *We say that f is compatible with g if for $\alpha = \min(|f|, |g|)$,*

$$f[\alpha] = g[\alpha].$$

Remark 1.1.2. We will write $\langle m_0, m_1, \ldots m_{k-1} \rangle$ for a finite sequence of length k.

We give more than enough to define the game interesting us, but these general definitions are useful in some enlarged studies of the present subject and will give us a better understanding of the logic of the trees we are about to define. However, for our concrete purpose, the reader can imagine $\beta = \omega$ since it will always be the case in the questions that we will treat further.

Definition 1.1.3 (Long trees)**.** *Given an ordinal β and a set A, a β-tree on A is a subset $T \subseteq A^{<\beta}$ such that*

$$\forall \tau, \sigma \in A^{<\beta}, \sigma \in T \textit{ and } \tau \subseteq \sigma \to \tau \in T.$$

Elements of a tree are called nodes and in the case above, if $\sigma = \tau^\frown \langle a \rangle$, for some $a \in A$, σ will be called a child node of τ and τ the (unique) mother node of σ.

1. *A function $f \colon \beta \to A$ is a (complete) path across (or a branch of) T if*

$$\forall \alpha < \beta, \;\; f[\alpha] \in T.$$

The set of branches of T is written $[T]$.

2. *A function satisfying the same condition but with domain some ordinal $\delta < \beta$ will be called a pseudo-path trough $[T]$.*

3. *The bouquet around a node σ in T, $[\![\sigma]\!]_T$, is the set $[\![\sigma]\!]_\beta \cap [T]$. We will omit the T subscript when the context is clear.*

4. *A terminal node (or leaf) of T is a node having no proper extension in T.*

5. *The tree T is said to be pruned if every node in it extends in a branch across it, if β is a limit ordinal that is, T contains no terminal node and there are no pseudo-paths of length $\delta < \beta$ for δ a limit ordinal. In other words, all the pseudo-paths we can follow through T are branches.*

6. *A subtree of T is a tree included in T. Given a node $\sigma \in T$, the offspring of σ is the following subtree*

$$T_\sigma := \{\tau \in T : \tau \textit{ is an initial segment or an extension of } \sigma\}.$$

7. *If β is infinite, the tree T is said to be perfect (or bushy) if it is pruned and any node in it has two proper incompatible extensions.*

In the scope of the exposition of these tree-related concepts let us prove the following simple lemma.

Lemma 1.1.4. *Let T be a β-tree on a set A, then if T is bushy the bouquet around any node of length $\alpha < |\beta|$ in T has cardinality $2^{|\beta|}$.*

Proof. Given a bushy tree T and a node $\sigma \in T$, we label its two distinct proper incompatible extensions by 0 and 1. By iterating this process from σ of length $\alpha < |\beta|$, we come up with a copy, up to the merging of some nodes, of the full binary ω-tree (that depicts figure 1.3). Then, since every pseudo-path in T is a complete path, we can extend our tree to a copy of the full $|\beta|$-tree, with the same process iterated a transfinite amount of time. Since this copy is contained in the offspring of any σ, 2^β is a lower bound for the cardinality of any bouquet in this β-tree. It is also clear that this is an upper bound, concluding our proof. □

Notice that even if β is a cardinal, this is not a sufficient condition since there could still be an infinite pseudo-path that is not a path in T but such that every node of it also extend in enough complete paths. Figure 1.1 depicts an example of a finite binary tree and of a finite tree on the natural numbers.

We now define the game as follows. Given a set A, Anais and Bruce want to play a two-player game. To do so, they are playing one after the other and their moves are elements of A. The rules of the game are described by an ω-tree T, containing all the finite sequences of moves playable in this setting. We will always suppose T to be nonempty

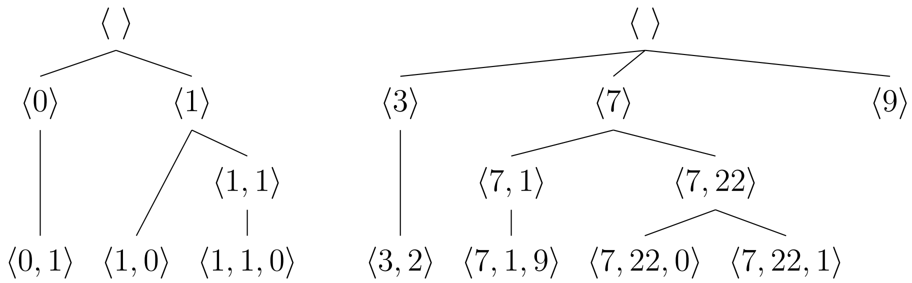

Figure 1.1: Examples of finite trees.

A:	a_0		a_2		a_{2n}			
				$\dots$			$\dots$	$(a_i)_{i<\omega} \overset{?}{\in} X$
B:		a_1		a_3		a_{2n+1}		

Playable move condition: $\forall n \ \langle a_0, a_1, \dots a_n \rangle \in T \subset A^{<\omega}$

Figure 1.2: An infinite game.

and pruned. Finally, they agree on a payoff set $X \subseteq A^\omega$. Anais plays first and wins if, after ω moves, the so-formed sequence belongs to the payoff set. Otherwise, victory belongs to Bruce. To emphasize the importance of the fact that a player plays first or second in a definition or a construction, we will rather use the denominations "player I" and "player II" for the convenience of notations and abstraction. Otherwise, we keep calling them Anais and Bruce for the convenience of readability and exemplification even if by convention –and gallantry– Anais will always play first and Bruce second. We give a representation of this game in figure 1.2.

For example, A can be the set of natural numbers ω but the playable moves are only 0 and 1. Then, if X is, let's say, the set of sequences that begin by 0, Anais has a winning strategy consisting only of playing 0 as her first move. The tree T of legal positions is the one presented in figure 1.3.

In general, a strategy for Anais is a way to respond to Bruce's moves by following the rules of the game. To define a strategy, notice that it is player II's turn to play after an odd-length sequence in T and conversely, player I's turn to play after an even-length node in T.

Definition 1.1.5 (player I's strategies)**.** *Given such a game $G(T, X)$, a* strategy S_{I} *for the first player will be defined as a subtree of T such that*

1. *Every even-length node has one unique child, the response of player* I*,*

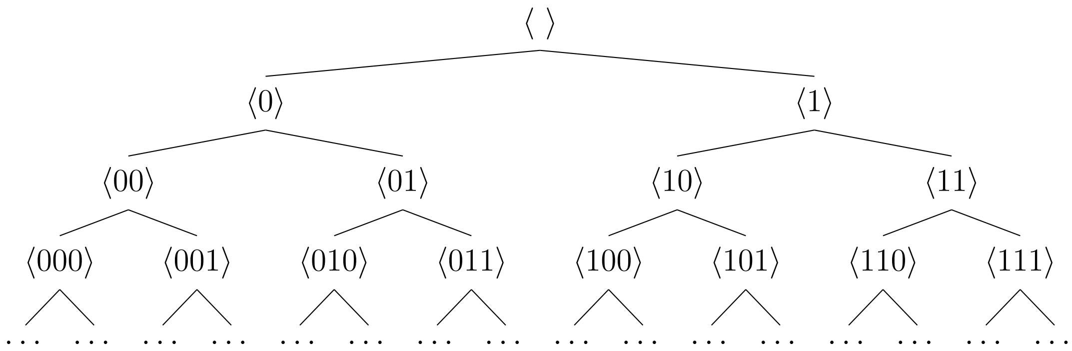

Figure 1.3: The infinite tree of binary playable moves.

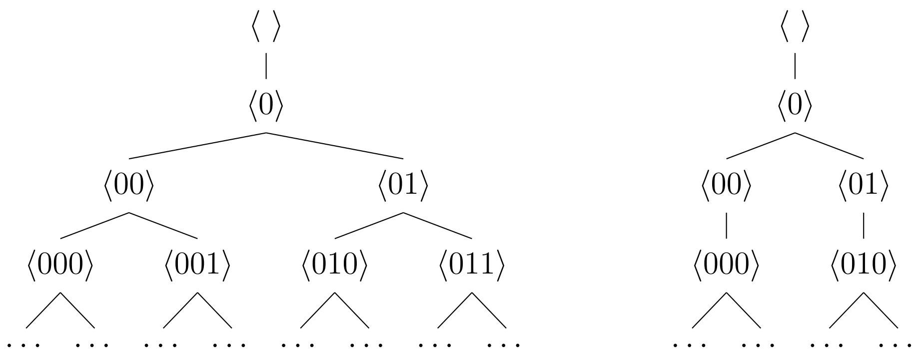

Figure 1.4: Ana's 0-quasistrategy and 0-strategy.

2. *Every child of an odd-length node that lies in T lies also in the strategy, i.e. the strategy doesn't restrain player* II*'s moves.*

We write the set of strategies for player I, $S_{\mathrm{I}}(T, X)$. *A strategy S is said to be winning if $[S] \subseteq X$, i.e. in any run of the game where player* I *plays according to S, she wins. In such a case, we say that any position during the play of the game is consistent with the strategy. A* quasistrategy *for her is defined in the same way, but the player* I*'s response has not to be unique.*

A strategy for Bruce will be defined *mutatis mutandis*. Note that, at any stage of the game the previous moves are common knowledge for both players. This is a game of perfect information. Equivalently, note that we can define a strategy for I (II) as a function from even-length (odd-length) sequences of T to the answer of I (II) in A such that this is a playable move at this point. In our last example, a (winning) quasistrategy for Anais is the left tree of figure 1.4. On the other hand, a (winning) strategy for Anais is depicted by the right tree of figure 1.4 which consists in always playing 0 regardless of Bruce's move.

We now present the proof –using ZFC axioms– of a simple known result, nevertheless a very foundational one. From now on $[T]$ will be endowed with topology of which a basis is formed by taking all the bouquets around its nodes.

Theorem 1.1.6 (Gale-Stewart)**.** *Let $G(T, X)$ be a game as defined above with T a nonempty pruned ω-tree on A and $X \subseteq [T]$ closed or open. Then either Anais or Bruce has a winning strategy.*

Proof. We prove that if X is closed, and Bruce has no winning strategy, then Anais has. The case with X being open is similar. We always assume the players are playing legal moves according to T.

We will say that an even-length position $p \in T$ –with Anais to play next– is *non-losing* if Bruce has no winning strategy in the game with T_p for rules and

$$X_p = \{x \in A^\omega \mid p^\frown x \in X\}$$

for payoff set. So $\langle\ \rangle$ is non-losing by assumption. Moreover, if p is non-losing, Anais can make a move a_{2n} such that, whatever move a_{2n+1} Bruce will be playing next, the position $p^\frown\langle a_{2n}, a_{2n+1}\rangle$ is still non-losing. Indeed the negation of this assumption asserts that p is losing.

The winning strategy S for Anais is then defined recursively. Let p be a position of length $2n$ reached by a play of the game consistent with her strategy already defined until there. By the preceding, Anais can choose one of the $a_{2n} \in A$ such that the next even-length position will still be non-losing. For doing this for an arbitrary T and A, we need to use CHOICE.

Take any $x \in [S]$, then if $x \notin X$, X being closed, there is a basic neighbourhood N containing x and disjoint from X. WLOG, we can assume N is of the form $[\![p]\!]$ for some even-length initial segment of x. Then, p is a losing position for Anais, a contradiction. □

This first theorem about the determinacy of these games is the reason why the latter are called "Gale-Stewart games".

Given a set A, the **axiom of determinacy** for A, written AD_A, states that whatever sets $T \subseteq A^\omega$ and X are, there will exist set that is a winning strategy either for Anais or for Bruce. We then say that the game $G(T, X)$ is determined. We always suppose that T satisfies the hypotheses of the preceding theorem.

This statement is of course way stronger than the Gale-Stewart theorem since the payoff sets X can be much more complicated than open or closed sets in the topology of $[T]$. We will only focus during this study on the case of countable games (move from ω), but the interested reader can find development about uncountable games in [19] or even class-sized games in [20]. Even for $A = \omega$ in which case we simply write AD, assuming the axiom of determinacy can yield quite strong theorems. To expose one of them we first introduce the following notions of descriptive set theory.

Definition 1.1.7 (Polish space)**.** *A topological space X is said to be* Polish *if*

1. *The space X is metrisable and complete for this metric,*
2. *There is some countable dense subset of X, that is, X is separable.*

Putting the discrete topology on A, the product topology makes of A^ω a metrisable complete space whose basic open sets are the ω-cylinders of finite sequences in A. We assume familiarity with the basics of topology (see for example [32]).

Lemma 1.1.8. *Given a set A then with the topology described above*

1. *A^ω is a metrisable complete space,*
2. *$A^{<\omega}$ is dense in A^ω,*

where $A^{<\omega}$ is seen as a subset of A^ω by completing every finite sequences with zeroes. In particular if A is countable then A^ω is Polish.

Proof. We define

$$\begin{aligned} d\colon A^\omega \times A^\omega &\to \mathbb{R}, \\ ((x_n)_{n<\omega}, (y_n)_{n<\omega}) &\mapsto \sum_{n<\omega} \frac{1-\delta(x_n, y_n)}{2^{n+1}}, \end{aligned}$$

as our metric on A^ω, its metric properties being straightforward from the definition of the Kronecker delta, $\delta(x_n, y_n)$ being equal to 1 if $x_n = y_n$ and 0 otherwise. Using geometric sums calculus, we observe the following properties. Given $x, y \in A^\omega$, $x[k] = y[k]$ iff $d(x, y) < 1/2^k$ for all $i < \omega$. Let us prove that (A^ω, d) is complete. Let $(x_n)_{n<\omega}$ being a Cauchy sequence of sequence in A, i.e.

$$\forall k \ \exists n_0 \ \forall m \geq n_0(k) \quad d(x_{n_0}, x_m) \leq \frac{1}{2^k}.$$

Thus by our preceding observation $x_{n_0(k)}[k] = x_m[k]$. And the sequence

$$\bar{x}(k) \coloneqq x_{n_0(k)}(k),$$

is the limit of $(x_n)_{n<\omega}$ in A^ω. We naturally (with respect to d) include $A^{<\omega}$ in A^ω by concatenating any finite sequence σ with an infinite sequence of zeroes. By definition the topology is generated by the family of open sets

$$\{\Omega_k^a : a \in A,\ k < \omega\}, \quad \text{where } \Omega_k^a \coloneqq \{(x_n)_{n<\omega} \subseteq A^\omega \mid x_k = a\}.$$

We now consider the basis consisting of all the ω-cylinders around a finite sequence. We have the following equalities on the open sets of the respective basis:

$$\Omega_k^a = \bigcup_{|\sigma|=k} [\![\sigma^\frown a]\!]; \qquad [\![\sigma]\!] = \bigcap_{a_i \in \sigma} \Omega_i^{a_i};$$

for any $k < \omega$, $a \in A$ and $\sigma = \langle a_0, a_1, \dots a_i, \dots a_k \rangle \in A^{<\omega}$. Hence we have proved that the metric, inducing the topology of the ω-cylinders by our first remark on the metric, also induces the product topology: we have proved point 1. It is also clear that any Ω_i contains a finite sequence, proving point 2. □

Therefore we speak about the Borel hierarchy of A^ω, which we define below.

Definition 1.1.9 (Borel hierarchy)**.** *On any topological space, we define inductively the following hierarchy of sets:*

1. *The $\underset{\sim}{\Sigma}_1^0$ sets are the open sets;*
2. *The $\underset{\sim}{\Pi}_1^0$ sets are the closed sets;*
3. *For any ordinal $1 < \alpha < \omega_1$, where ω_1 denotes the first uncountable ordinal, a set Y is $\underset{\sim}{\Sigma}_\alpha^0$ if there are*

$$A_0, A_1, \dots A_n, \dots \quad \textit{such that} \quad Y = \bigcup_{n<\omega} A_n,$$

   with each A_n being $\underset{\sim}{\Pi}_{\alpha_n}^0$ for some $\alpha_n < \alpha$;

4. *A set is $\underset{\sim}{\Pi}_\alpha^0$ if it is the complement of a $\underset{\sim}{\Sigma}_\alpha^0$ set.*

We also call $\underset{\sim}{\Delta}_\alpha^0$ sets the ones being both $\underset{\sim}{\Sigma}_\alpha^0$ and $\underset{\sim}{\Pi}_\alpha^0$ sets.

A set will be called Borel if it is $\underset{\sim}{\Sigma}_\alpha^0$ for some $\alpha < \omega_1$. Indeed, no new sets are defined beyond countable ordinals. These sets are usually defined as the minimal $\underset{\sim}{\Sigma}$-algebra containing the open sets. This hierarchy shows how to construct them, using a transfinite process. For the last definition, we use the Baire space, the universal Polish space

$$\mathcal{N} := \omega^\omega,$$

endowed with the product topology on the copies of the discrete spaces ω. We call it universal since every Polish space can be viewed as a closed subset of $\mathcal{N}$ (see first chapter of [40]). Precisely, every perfect Polish space is Borel isomorphic with the Baire space, where "perfect" means "without isolated points" and a Borel isomorphism has the property the preimage of a Borel set is still Borel. Using continued fractions like in [22], we can show that this space is homeomorphic to the irrational numbers of $\mathbb{R}$, we will hence identify it to the real numbers very often.

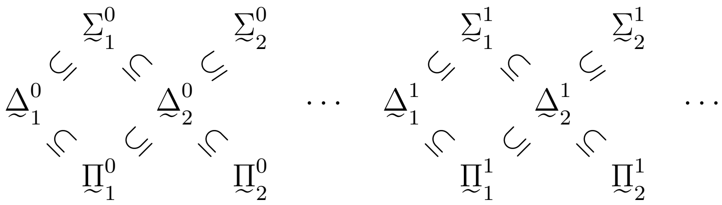

Figure 1.5: The Borel and projective hierarchies.

Definition 1.1.10 (Projective hierarchy)**.** *On any topological space X, we define inductively the following hierarchy of sets:*

1. *A set is $\boldsymbol{\Sigma}^1_1$ or analytic if it is the range of a continuous function $f\colon \mathcal{N} \to X$;*
2. *A set is $\boldsymbol{\Pi}^1_1$ or coanalytic if it is the complement of an analytic set;*
3. *For any natural number $1 < n < \omega$, a set is $\boldsymbol{\Sigma}^1_n$ if it is the range of a continuous function $f\colon Y \to X$, with Y being a $\boldsymbol{\Pi}^1_k$ set for some $k < n$;*
4. *A set is $\boldsymbol{\Pi}^1_n$ if it is the complement of some $\boldsymbol{\Sigma}^1_n$ set.*
5. *A set is $\boldsymbol{\Delta}^1_n$ if it is both a $\boldsymbol{\Sigma}^1_n$ and a $\boldsymbol{\Pi}^1_n$ set.*

A set will be called projective if it is $\boldsymbol{\Sigma}^1_n$ for some $n < \omega$. Since we will work with the Baire space in the following, we will only consider such hierarchies in Polish spaces.

Remark 1.1.11. From now on, when stating a theorem or a proposition, we will often indicate between parentheses, inside of which theory we claim the result to be provable. This indication could be preceded by the name of the theorem.

The following theorem gives the link between the two hierarchies of Borel and projective sets.

Theorem 1.1.12 (Suslin, ZF)**.** *For any Polish space, the Borel sets are exactly the sets that are both analytic and coanalytic, that is, the $\boldsymbol{\Delta}^1_1$ sets.*

Based on this last theorem, we can unify the two hierarchies into a scale of complexity of definable sets in Polish spaces. Furthermore, we can show that this scale is monotone as depicted in figure 1.5: a standard result proven for instance in [40].

A set $X \subseteq \mathbb{R}$ is Lebesgue measurable if for each $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \cap \bar{X}), \tag{1.1}$$

where μ^* stands for the outer measure of a set of real numbers, i.e. the infimum on the volume of coverings using intervals. For example any union of intervals or even any analytic set (see definition 1.1.10) is Lebesgue measurable. In such a case, we write μ for taking the measure of the set, which is equal to its outer measure. More details about it, including the proposition 1.1.13, can be found in [23], see also [7] for a more specialized book.

Although $\underset{\sim}{\Sigma}^1_1$ sets are Lebesgue measurable, as Lusin said, "one does not know and one will never know" about the measurability of $\underset{\sim}{\Sigma}^1_2$ sets. This is where the hypothesis about the existence of large cardinals comes into play. If we want to get more results about measurability, we need to assume more axioms than those we already supposed by working in ZFC.

Proposition 1.1.13 (Jan Mycielski and S. Świerczkowski (1964), $\mathsf{ZF}+\mathsf{AD}$)**.** *Every set of real numbers is Lebesgue-measurable.*

The proof of this very early result, in a paper by Jan Mycielski and S. Świerczkowski [43] and follows from the following lemma, also provable assuming AD. We call a set *null* if its outer measure is 0.

Lemma 1.1.14. *Let S be a set of real numbers such that every measurable $Z \subseteq S$ is null, then S is null.*

To make use of the axiom of determinacy we define the covering game.

Definition 1.1.15 (The covering game)**.** *Let $S \subseteq \mathbb{R}$ and $\epsilon > 0$. The* covering game *of S is defined as follows.*

Anais plays binary moves and Bruce natural numbers, this leads to a tree $T \subsetneq \omega^{<\omega}$ defining the rules of the game.

Given $f \in 2^\omega$, we define the real

$$\bar{f} = d(0, f), \tag{1.2}$$

with d as in the proof of lemma 1.1.8. For each $n < \omega$ we define

$$K_n = \left\{ G \subseteq \mathbb{R} \mid G = \bigcup_{j<k} I_j \text{ for some } k < \omega \text{ and } \mu(G) \leq \epsilon/2^{2(n+1)} \right\},$$

where the I_j's are intervals with rational endpoints, hence K_n is countable and we denote by $(G^n_k)_{k<\omega}$, an enumeration of K_n.

The payoff set $X \subseteq \mathcal{N}$ is defined by the sequences f such that

$$a := \bar{f}_{|\mathrm{I}} \in S \qquad \text{and} \qquad a \notin \bigcup_{n=0}^{\infty} G^n_{f_{|\mathrm{II}}(n)},$$

where $f_{|\mathrm{I}}(n) := f(2n)$ and $f_{|\mathrm{II}}(n) := f(2n+1)$.

Note that the operation 1.2 is surjective on $[0,1]$ (it suffices to use the infinite binary expansion of these numbers).

Proof of lemma 1.1. Because of the definition of outer measure, we can assume S is contained in a finite interval, WLOG we assume $S \subset [0,1]$. We first show that Anais has no way to get a winning strategy while playing the covering game.

Consider a winning strategy Σ for player Anais. We can extract from it a function

$$f \colon \mathcal{N} \to \mathbb{R};$$
$$b \mapsto \bar{x}_{|\mathrm{I}},$$

where x is the result of a play of the covering game when Anais follows Σ and $x_{|\mathrm{II}} = b$. The function f is the tool that helps us to verify Anais effectively wins, given any play b of Bruce. It is easy to see that f is continuous, hence $Z \coloneqq f(\mathcal{N})$ is analytic and then measurable. Since Σ is winning, $Z \subseteq S$ and then, by hypothesis, is null. However, a null set can be covered by a countably infinite union

$$\bigcup_{n=0}^{\infty} H_n \qquad \text{such that} \qquad \forall n < \omega \;\; H_n \in K_n.$$

This way, playing $(b_i)_{i<\omega}$ such that $G^n_{b_i} = H_i$ for all i, Bruce can win while Anais is still following her strategy, a contradiction.

Now we make use of AD to get a winning strategy for Bruce since the covering game is determined. Consider such a winning strategy τ for Bruce. For each finite binary sequence of Anais' moves $s = \langle a_0, a_1, \ldots a_n \rangle$, following τ leads Bruce to play some b_s as $(n+1)$st answer to the playing of Anais. Lets denote by G_s, $G^n_{b_s} \in K_n$. By surjectivity of the operation 1.2 we can associate to each $a \in S$ the set B_a of its infinite binary expansions. Since τ is winning, for every $a \in S$

$$a \in \bigcup \{G_s \colon s \text{ is in initial segment of any } f \in B_a\} \qquad \text{and then,}$$
$$S \subseteq \bigcup \{G_s \colon s \in 2^{<\omega}\} = \bigcup_{n=1}^{\infty} \bigcup_{s \in 2^n} G_s.$$

By definition, for every $n \geq 1$, and s of length n, $\mu(G_s) \leq \epsilon / 2^{2n}$, hence

$$\mu(\bigcup_{s \in 2^n} G_s) \leq \frac{\epsilon}{2^{2n}} \cdot 2^n = \frac{\epsilon}{2^n}.$$

It follows then that

$$\mu^*(S) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, S is null. ∎

Proof of proposition 1.1.13. Let $X \subseteq \mathbb{R}$. It follows easily from the definition of the outer measure that there exists a measurable set A containing X such that each measurable set $Z \subseteq A \cap \bar{X}$ is null. Then, by the preceding lemma $\mu^*(A \cap \bar{X}) = 0$. Given any $Y \subseteq \mathbb{R}$ we now show equation 1.1 is verified. Note that we always have

$$\mu^*(Y) \leq \mu^*(Y \cap X) + \mu^*(Y \cap \bar{X})$$

so we only have to prove the other equality:

$$\begin{aligned} \mu^*(Y \cap X) + \mu^*(Y \cap \bar{X}) &= \mu^*(Y \cap X) + \mu^*(Y \cap \bar{X} \cap A) + \mu^*(Y \cap \bar{X} \cap \bar{A}) \\ &\leq \mu^*(Y \cap A) + 0 + \mu^*(Y \cap \bar{A}) \\ &= \mu(Y), \end{aligned}$$

by the measurability of A, the result of the lemma and the obvious monotonicity of the outer measure. □

Consequently, regarding some non-measurable sets of reals constructible with the use of Choice, (see for example [21]), we have ZF proving the inconsistency of Choice with AD. However, it is worth noticing we can prove the countable version of the axiom of choice with AD as it was proved first by Mycielski (see [23]). We will encounter a more precise example of this phenomenon in the following section. Unlike this first very powerful statement, we will be interested in this study in determinacy for "definable" sets like those we just exposed above. Several properties on the concerned class of definable sets are then implied by determinacy axioms in various fields of study like in descriptive set theory as we already mentioned, but also topology [6], Polish group actions [4], harmonic analysis with sets of uniqueness [27], orbit equivalence [28], dynamical systems [14], ...

For example, as we have already proved, if we restrict the axiom of determinacy to payoff sets that are open or closed, it is a consequence of ZFC (and thus not contradictory with choice). We can prove better as the following theorem shows, which we won't prove here, emphasises the necessity of Replacement, added by Fraenkel to the original set theory of Zermelo, and Power Set as discussed in the introduction of [5], where one can find a study of the limits of the provability of determinacy in ZFC, and in the book of Martin [35] which initially proved the theorem in [34]. By the limits of determinacy, we mean the largest definable class of sets with which the determinacy axiom is provable.

Theorem 1.1.16 (Martin (1975), Borel determinacy, ZFC)**.** *Let T be a nonempty pruned tree on A and let $X \subseteq [T]$ be Borel. Then $G(T, X)$ is determined.*

On the other hand, one only needs a weaker version of Choice, namely the axiom of dependent choice, to prove this result.

Despite this result being somewhat optimal in ZFC, the study of determinacy for projective sets is a vast source of axioms for extensions for ZFC, closely related to large cardinal hypotheses. For the latter, we refer to [36, 54].

Let us finish this section with a point of view from *infinitary logic*, roughly speaking, logic with infinite sentences, on determinacy. Observe that $G(T, X)$, with $T = A^{<\omega}$, is a win for player I (i.e. she has a winning strategy) if and only if the infinite sentence

$$(\exists a_0)(\forall a_1)(\exists a_2)(\forall a_3)\cdots \quad \langle a_n\rangle_{n<\omega} \in X$$

holds. The same way, G is a win for player II iff

$$(\forall a_0)(\exists a_1)(\forall a_2)(\exists a_3)\cdots \quad \langle a_n\rangle_{n<\omega} \notin X$$

holds. Consequently, the determinacy of G is equivalent to the infinite sentence

$$\neg[(\exists a_0)(\forall a_1)\cdots \quad \langle a_n\rangle_{n<\omega} \in X] \leftrightarrow (\forall a_0)(\exists a_1)\cdots \quad \langle a_n\rangle_{n<\omega} \notin X.$$

This can be seen as a natural generalisation of the finite classical logical law

$$\begin{gathered}\neg[(\exists a_0)(\forall b_0)\cdots(\exists a_k)(\forall b_k)\phi(a_0, b_0, \ldots a_k, b_k)] \\ \leftrightarrow \\ (\forall a_0)(\exists b_0)\cdots(\forall a_k)(\exists b_k)\neg\phi(a_0, b_0, \ldots a_k, b_k)\end{gathered}$$

for every formula ϕ, which by the way asserts the determinacy of games of length k. We can see ϕ as the defining formula of some subset X in A^ω. For instance, X could be a Boolean combination of bouquets of nodes of length k. Thus, the determinacy of Borel games teaches us that we can extend this classical logical law to an infinitary one, if ϕ defines a Borel subset of A^ω. On the other hand, Choice imply that this rule is false for arbitrary set X, and ZF even prove that there are undetermined uncountable sets (see [23] for the construction of such non-determined games).

1.2 Second-Order Arithmetic

Among the provable theorems within ZFC some theorems require much of the strength of the axioms such as Borel determinacy. On the other hand, theorems such as the intermediate value theorem or the Bolzano-Weierstrass theorem don't require powerful mathematical principles, even though they may use also the same kind of axioms for set theory in their proof. Consequently, it is difficult to see, *a priori*, how we could classify these theorems according to the difficulty of proving them. By this "difficulty" we mean which are the minimal mathematical principles that must be assumed to prove them. A satisfying answer to this questioning would be to dispose of one or multiple hierarchies of axiom systems, to assess the provability of the mathematical theorems and therefore lead an analysis of their logical strength.

This study of the classification of mathematical theorems is the idea of reverse mathematics first introduced by Friedman ([15, 16]) and Simpson, who wrote a reference book on the subject in collaboration with a lot of other researchers on the subject, [48]). A convenient framework for this study is the two-sorted language of second-order arithmetic, the weakest language in which we can implement most ordinary mathematics. It is the outcome of the idea of "encoding" mathematical objects through natural numbers and sets of natural numbers, over which two kinds of variables will range.

The idea of a code for a mathematical object already exists in the standard setting of set theory, since we want to depict most accurately a concept through a simulacrum expressed by the means of the chosen mathematical language. We cannot say this is the real object, would it only exist! Considering the Von Neumann representation of natural numbers, for example, it wouldn't make sense to pretend that zero is, as a concrete entity, the abstract empty set. This is just a code, deemed to simulate a concept: the idea of what zero should be. In the scope of second-order arithmetic, because it is more unusual, we will speak of the code of a mathematical object with the same meaning. For instance, any countable object can be coded as a subset of natural numbers. However, since it has *a priori* no intrinsic concrete existence, this code is not more a code than its usual representation in set theory.

We will distinguish between the numerical variables $m, n, k, l, \dots$ and set variables $X, Y, Z, S, \dots$ by use of capitalisation. The language of second-order arithmetic, familiar since early mathematical education, is $L_2 = \{+, \cdot, 0, 1, <, \in\}$, where the function and relation symbols have each their standard definition.

Definition 1.2.1 (Z_2)**.** Full second-order arithmetic *theory* Z_2 *is the* L_2*-theory consisting of the axioms of* discrete ordered semi-ring

$m + 1 \neq 0$	$m \cdot 0 = 0$
$m + 1 = n + 1 \to m = n$	$m \cdot (n + 1) = (m \cdot n) + m$
$m + 0 = m$	$\neg(m < 0)$
$m + (n + 1) = (m + n) + 1$	$m < n + 1 \leftrightarrow (m < n \vee m = n)$

that we will call the basic axioms *plus the set induction axiom*

$$(0 \in X \wedge \forall n \ (n \in X \to n + 1 \in X)) \to \forall n \ (n \in X)$$

and the full *comprehension scheme*

$$\exists X \ \forall n \ (n \in X \leftrightarrow \phi(n)),$$

where X does not occur freely in ϕ.

This last condition prevents us from contradictions generated by auto-references like $X = \{n : n \notin X\}$. Note also that the full comprehension scheme doesn't give rise to contradictions like in naive set theory. Indeed all the sets here are included in the set $X = \{n : n = n\}$, which we will call $\mathbb{N}$ and plays the role of the universe for the set of second-order arithmetic. Therefore we can view this last scheme as an instance of separation for the set of natural numbers. This works well because we evolve in a two-sorted language, we could also deal with an unbounded comprehension scheme, with the variable "n" ranging on set variables in L_{Set}, provided that we introduce a second type of variables, commonly denoted "classes", deemed to be collections of sets. In this particular setting, all classes would be included in $X = \{n : n = n\}$, the class of all sets.

Remark 1.2.2. The induction axiom with the full comprehension scheme proves the full second-order induction scheme, stated for any L_2-formula $\phi(n)$,

$$(\phi(0) \wedge \forall n\ (\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n\ \phi(n).$$

This could still be the case, even without *full* comprehension scheme, in any model whose first-order part is the standard set of natural numbers ω. In general, we refer to the set of natural numbers in a given L_2-structure as $\mathbb{N}$. The latter could contain non-standard natural numbers, unlike ω.

Definition 1.2.3 (ω-model)**.** *Given an L_2-theory T, an ω-model for T is a model whose first order part is the standard set of natural numbers, ω.*

We are now able to implement the reverse mathematics project. The aim is now to restrain the complexity of the formulae one can use in the comprehension scheme to create new sets, a crucial construction in any proof. This way, we will dispose of a hierarchy of sub-theories of second-order arithmetic of increasing strength. First, we have to define what we mean by "complexity".

Definition 1.2.4 (Complexity hierarchy for L_2-formulae)**.** *Given a formula $\phi(n)$, a* bounded numeric quantification *of ϕ is a formula of the form*

$$\forall n\ (n < m \rightarrow \phi(n)) \qquad or \qquad \exists n\ (n < m \wedge \phi(n)).$$

Thus, we call Δ^0_0 the class of formulae of L_2 containing the atomic formulae and closed under $\neg$, $\wedge$ and bounded numeric quantification. We then define the following hierarchy of classes of formulae:

1. *The Σ^0_1 formulae are of the form $\exists n \theta(n)$ with θ a Δ^0_0 formula;*
2. *The Π^0_1 formulae are of the form $\forall n \theta(n)$ with θ a Δ^0_0 formula;*
3. *For any natural number $1 < k < \omega$, a formula is Σ^0_k if it is of the form $\exists n \theta(n)$ with θ a Π^0_{k-1} formula;*

4. *For any natural number* $1 < k < \omega$*, a formula is* Π^0_k *if it is of the form* $\exists n \theta(n)$ *with* θ *a* Σ^0_{k-1} *formula.*

Finally, for any $1 \leq k < \omega$*, a formula is* Δ^0_k *if it is equivalent to two formulae* ϕ *and* ψ*, which are respectively* Σ^0_k *and* Π^0_k*. This latter equivalence depends on the model or the theory in which we are reasoning. We say that a formula is* arithmetical *if it contains no set quantifiers. Moreover, we define* Σ^1_k*,* Π^1_k *and* Δ^1_k *formulae similarly, where the domain of the quantifier are set variables and arithmetical formulae play the role of* Δ^1_0 *formulae.*

It is no coincidence that this definition looks very similar to definitions 1.1.9 and 1.1.10. We can define the finite steps of the Borel and projective hierarchy of the Baire space $\mathcal{N}$ by Σ^0_k and Σ^1_k formulae. It turns out that we can speak of any Polish space in second-order arithmetic even if we cannot encode uncountable sets in general.

Taking any countable set A, provided that we dispose of an injection from A to $\mathbb{N}$, we can code A as a subset of $\mathbb{N}$. For example, this is the case for the set of couples of natural numbers, via the pairing function

$$\begin{aligned} &f\colon (m,n) \mapsto (m+n)^2 + m, \quad \text{which is injective since} \\ &\forall m \forall n \quad f(m,n) < f(m,n+1) < f(m+1,n) < f(m+1,n+1). \end{aligned}$$

We will therefore identify $\mathbb{N}\times\mathbb{N}$ to its image by the pairing function and write $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{N}$. This way we can also encode any subtree of $\mathbb{N}^{<\mathbb{N}} \subseteq \mathbb{N}$. Finite sequences, and in particular couples, of natural numbers are indeed very useful in the process of coding (and we may need to include non-standard finite sequences if the considered model is not an ω-model).

Remark 1.2.5. Note that we don't have an extensionality axiom in second-order arithmetic. The equality relation is only formally used between numerical variables. This is convenient since this way we can speak of a set of natural numbers as well as some countable set coded by this same set at the same time without identifying them, despite the fact they have the same elements. That's why it is not a problem to write $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{N}$.

Following Hilbert's programme for the arithmetisation of mathematics, we can code most of ordinary mathematics. For example, any element in a Polish space is the limit of some Cauchy sequence (hence equality between them will be an equivalence relation). Moreover, an open set will be coded as a subset U of $A \times \mathbb{Q}^+$, where A denotes a dense part of the Polish part. We should think of this code as the collection of the open balls contained in the open sets. We will then say that x is an element of this open set iff

$$\exists (a,r) \in U \text{ such that } \mathrm{d}(a,x) < r,$$

where d is the metric on the Polish space. We see that this condition is indeed Σ^0_1. Finally given a sequence $A_0, A_1, \ldots\ A_n, \ldots$ of subsets of natural numbers, we can define their union and intersection as the respective sets

$$\bigcup_n A_n = \{a : \exists m\ (a \in A_m)\} \quad \text{and} \quad \bigcap_n A_n = \{a : \forall m\ (a \in A_m)\}.$$

Therefore, we see that the link with the Borel hierarchy is obvious. From now on we think alternatively and equivalently of a set by "the set itself" and the formula defining it (a code for this set). Even if we cannot talk about arbitrary uncountable subsets of a Polish space, it is often possible to code such sets which are definable. Concerning the projective hierarchy, we can show that given a code A for an analytic set and a code X for an element of the concerned Polish space, the formula $X \in A$ is Σ^1_1. The latter enables us to finalise our analogy with the previously defined hierarchies. We can now state the natural translation of the axiom of determinacy inside second-order arithmetic.

Definition 1.2.6 (Σ^0_k determinacy)**.** *For all $1 \leq k < \omega$ we denote by Σ^0_k-*Det *the axiom stating that for any tree $T \subseteq \omega^{<\omega}$ and any Σ^0_k formula $\phi(X)$ that is deemed to describe the payoff set $\tilde{X} \subseteq \omega$ of corresponding complexity, the game $G(T, \tilde{X})$ is determined.*

The determinacy axiom for other complexity classes is of course defined similarly. Let us now define some major sub-theories of second-order arithmetic.

Definition 1.2.7 (RCA_0, ACA_0, Π^1_1-CA_0)**.** *We define three L_2-sub-theories of Z_2. They each consist of their basic axioms and the induction axiom for sets, plus a characteristic occurrence of the comprehension scheme.*

1. *The theory of* recursive comprehension*, RCA_0, consists of the basic axioms plus*

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X\ \forall n\ (n \in X \leftrightarrow \phi(n)),$$

   with ϕ and ψ being Π^0_1 and Σ^0_1 respectively; the comprehension scheme for, somehow, the Δ^0_1 formulae where X does not occur freely endowed with

$$(\phi(0) \land \forall n\ (\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n\ \phi(n),$$

   the induction scheme for formulae ϕ that are Σ^0_1.

2. *The theory of* arithmetical comprehension*, ACA_0 consists of the basic axioms plus the comprehension scheme for formulae ϕ that are arithmetical and X does not occur freely.*

3. *The theory of Π^1_1 comprehension, Π^1_1-CA_0, consists of the basic axioms plus the comprehension scheme for formulae ϕ that are Π^1_1 and X does not occur freely.*

The subscript 0 in such an acronym for a theory of L_2 is there to remind us of the fact that the induction scheme axiom for formulae is *a priori* restricted as previously discussed in remark 1.2.2.

Moreover, note that the formula used in the comprehension scheme, ϕ, even if arithmetical, may contain free set variables other than X. They should be interpreted as parameters for the defining formula.

Remark 1.2.8. Π^1_1-CA_0 proves the Σ^1_1 comprehension scheme. Consider any Σ^1_1 formula $\psi(n)$, then $\neg\psi(n)$ is Π^1_1 and so Π^1_1-CA_0 proves the existence of $\bar{X} = \{n : \neg\psi(n)\}$. Using Δ^0_0 comprehension we now show the existence of $X = \{n : n \notin \bar{X}\}$.

It turns out that most undergraduate mathematics can be implemented in the theory of arithmetical comprehension. For many theorems, even just recursive comprehension is sufficient while some others require Π^1_1 comprehension. We often denote them as formal systems to emphasize their potential use as a metatheory, endowed with a recursive construction of a formal language. Essentially, the soundness theorem and a version of Gödel's completeness theorem are provable in RCA_0. This allows us to develop mathematical logic in a reasonable perspective inside the theory –and consequently any model of– RCA_0. For this reason, it is even more relevant to call them formal systems, accordingly to their ability to simulate a large range of mathematics, including (weaker) theories of L_2 and some of their models.

Example 1.2.9. Recursive comprehension proves the intermediate value theorem. Arithmetical comprehension proves Bolzano-Weierstrass's theorem. Π^1_1 comprehension proves Cantor-Bendixson's theorem.

To give an example more related to our main topic, ACA_0 it is not strong enough to prove open determinacy in the Baire space. Despite this fact, we can still get such a result if we restrain ourselves to the Cantor space, 2^ω, or games with a binary choice. This is presented in the following result, where the "$*$" superscript is used to remind us of the restriction of our playground.

Theorem 1.2.10 (RCA_0)**.** $\mathsf{ACA}_0 \leftrightarrow (\Sigma^0_1)_2$-$\mathsf{Det}^*$.

Here, we denote by a $(\Sigma^0_1)_2$ set, the difference of two open sets. This definition will be generalized in by 3.1.5. In the paper of Nemoto and Tanaka [44], where this theorem comes from, it is proved that RCA_0, the axiom scheme of arithmetical comprehension implies determinacy of $(\Sigma^0_1)_2$-Det^*. Moreover, taking the theorem as an axiom in the same setting, we can prove back the scheme of arithmetical comprehension. It is from this characteristic method that the name "reverse mathematics" comes. Inside reverse mathematics, we say that arithmetical comprehension is the appropriate set of axioms to prove that $(\Sigma^0_1)_2$ sets are determined.

Despite this theorem, arithmetical comprehension is not sufficient to prove determinacy results about the Baire space, even for clopen payoff sets. Looking at the proof of theorem 1.1.6 about open or closed sets, we asked Anais to play the strategy defined by the following fact:

> "Anais can choose one of the $a_{2n} \in A$ such that the next even-length position will still be non-losing."

At first sight, it seems that this technique could be used in arithmetical comprehension. Moreover, we no longer need to use CHOICE, since she can always find the smallest possible natural number. However, an (even) position p, i.e. an (even-length) sequence in the tree T, is said to be *non-losing* for player I if player II has no winning strategy from then on, i.e.

$$\forall S_{\text{II}} \ (S_{\text{II}} \text{ is a strategy for player II in } G(T_p, X_p) \rightarrow \exists x \in \ [T_p] \cap X).$$

So to define the set of the future *non-losing* positions, we should use an instance of comprehension allowing us to use formulae with set quantifiers!

Remark 1.2.11. We can easily show that the set of *non-losing* positions in the tree has a Π^1_2 definition.

However, such a proof would not be optimal from a reverse mathematics point of view. Indeed, we only need to use a weaker additional axiom, the one of recursive recursion, as suggested in an exercise presented in [26, 20.2]. We will now explain the idea of transfinite recursion in second-order arithmetic, using this sketch of proof for determinacy for closed games as an illustration. To this end, let T be a nonempty pruned tree on $\mathcal{N}$ and let $X \subseteq [T]$ be closed. Thus $X = [S]$ for S a subtree of T. First, let us define a key notion of transfinite recursion.

Definition 1.2.12 (Well orderings)**.** *For any relation Y, we denote by* field(Y) *the set of i such that (i, j) or (j, i) is in Y for some j. A reflexive relation $Y \subseteq \mathbb{N} \times \mathbb{N}$ is a* well-ordering *and we write $WO(Y)$ if*

1. *The relation Y is* anti-symmetric, transitive *and* total*, which we write* $\text{LO}(Y)$*,*
2. *The relation Y is* well founded*, that is it has no infinite descending sequence, i.e.*

$$\neg(\exists (x_n)_{n\in\mathbb{N}} \subseteq \text{field}(Y) \ \forall n \ (x_{n+1}, x_n) \in Y)$$

which we write $\text{WF}(Y)$*, a Π^1_1 condition.*

Remark 1.2.13. Note that the proper existence of the field of Y needs at least Σ^0_1-comprehension. However, if Y is reflexive, it can be described in the simplest of ways as the set of i such that $(i, i) \in Y$ and we don't need that anymore.

Suppose Y is a code for a countable ordinal, that is, Y is a countable well-ordering. Given an arithmetical formula $\theta(n, W)$, we want to associate to each j in the field of Y the sets:

$$W^j = \{(m, i) : i <_Y j \wedge m \in W_i\} \quad \text{and} \quad W_j = \{n : \theta(n, Y^j)\}. \tag{1.3}$$

According to our illustration, we want to consider the arithmetical formula

$$\theta(\sigma, W) = \sigma \in T \setminus S \vee \forall m \exists n \exists i \ (\sigma^\frown\langle m, n\rangle, i) \in W,$$

where we should add that σ must be a even-length sequence (coded in $\mathbb{N}$) and m must be a playable move according to T. This construction is made in [48, V.8.2] for proving open determinacy with (countable) transfinite recursion. In our case, we would like to define this way, for each countable ordinal β, the following sequence of sets:

$$\begin{aligned} &\sigma \in W_0 \leftrightarrow \sigma \text{ is an even-length sequence } \wedge \sigma \in T \setminus S; \\ &\sigma \in W_\beta \leftrightarrow \forall m \ (\sigma^\frown\langle m\rangle \in T \rightarrow \exists n \ (\sigma^\frown\langle m, n\rangle) \in \bigcup_{\alpha<\beta} W_\alpha). \end{aligned}$$

If σ belongs to W_0, the game is obviously a win for player II since we will fall out of $X = [S]$. If σ belongs to W_β, it means that whatever move player I can play, player II can always play a move such that the following position is in W_α for $\alpha < \beta$. Then we can show that player II has a winning strategy in $G(T, X)$ iff $\emptyset \in \bigcup_{\beta<\omega_1} W_\beta$. Let us thus define the following predicate.

Definition 1.2.14. *Let $\theta(n, W)$ be any formula. We define $\mathrm{H}_\theta(Y, W)$ to be the formula which says:*

1. *The set Y codes a linear ordering;*
2. *W is the set of pairs (m, j) such that j is in the field of Y and $\theta(n, W^j)$ holds, where W^j is defined as in (1.3).*

Note that if θ is endowed with parameters or is arithmetical, then so is $\mathrm{H}_\theta(Y, W)$.

It is provable in ACA_0 that for any given formula θ and coded well-ordering Y, if some W satisfies the above formula, then it is the unique set with this property.

Definition 1.2.15 (ATR_0)**.** *The L_2-sub-theory of Z_2 of the* axiom of transfinite recursion *consists of the axioms of ACA_0, together with the axiom scheme:*

$$\mathrm{WO}(Y) \rightarrow \exists W \ \mathrm{H}_\theta(Y, W),$$

where $\mathrm{WO}(Y)$ stands for "Y codes a countable well order" and θ is arithmetical.

The notion of well-orderings is also used to encode countable ordinals.

Definition 1.2.16 (β-models)**.** *Given any L_2-theory T, a β-model M of T is an ω-model of T such that for every Π^1_1 formula $\phi(X)$,*

$$\forall X \in M, \quad \phi(X) \leftrightarrow M \models \phi(X).$$

Note that the standard ground-model behind such a model of T contains full Z_2, where we check whether $\phi(X)$ is true. We can think of it as "the real world". While the idea behind an ω-model is to have the standard natural numbers, a β-model has thus standard natural numbers and ordinals. Indeed, we have seen in definition 1.2.12 that being a code for an ordinal is a Π^1_1 condition.

In the same way that any ω-model satisfies induction for all formulae of L_2, it can be shown that any β-model satisfies *transfinite induction* for all formulae of L_2 and that this imply furthermore that it satisfies ATR_0.

The following theorem states that open determinacy for games in ω is equivalent, in the sense of reverse mathematics, to the axiom of transfinite recursion. Hence, ATR_0 is the right set of axioms to prove it. This is done in [48] and [49], but we essentially already presented how to lead the proof, in our illustration of transfinite recursion.

Theorem 1.2.17 (ACA_0)**.** $\mathsf{ATR}_0 \leftrightarrow \Sigma^0_1\text{-}\mathsf{Det}$.

One direct application of Σ^0_1 determinacy is the ability to prove some form of the axiom of choice in second-order arithmetic.

Definition 1.2.18 (Axiom of choice scheme)**.** *For any $0 \leq k < \omega$, the scheme of Σ^1_k choice is*

$$\forall n\ \exists Y\ \eta(n, Y) \to \exists Z\ \forall n\ \eta(n, (Z)_n),$$

where $\eta(n, Y)$ is any Σ^1_k formula in which Z does not occur. We are using the notation

$$(Z)_n = \{i : (i, n) \in Z\}.$$

We write it $\Sigma^1_k\text{-}\mathsf{AC}_0$.

For example, the scheme of Σ^1_1 choice asserts the validity of the use of Choice for countable collections of nonempty analytic sets of real numbers in second-order arithmetic. Here Z would code the choice function. We now state the application of Σ^0_1 determinacy.

Theorem 1.2.19. $\mathsf{ATR}_0 \vdash \Sigma^1_1$ *axiom of choice.*

Before tackling the proof itself, we need some classical tools to handle the Σ^1_1-formulae.

Lemma 1.2.20 (ACA_0). *Let $\phi(X)$ be a Σ^1_1-formula. Then we can find an arithmetical (in fact Σ^0_0) formula $\theta(\sigma,\tau)$ such that*

$$\forall X,\ (\phi(X) \leftrightarrow \exists f\ \forall m,\ \theta(X[m], f[m])).$$

Here f ranges over total functions $\mathbb{N} \to \mathbb{N}$, i.e. their domain is all the natural numbers. Also

$$X[m] = \langle \xi_0, \xi_1, \dots, \xi_{m-1} \rangle$$

where $\xi_i = 1$ if $i \in X$ and 0 otherwise. Note that $\phi(X)$ may contain free variables other than X. If so, then $\theta(\sigma,\tau)$ contain them too.

We refer to [48] for proof of this classical result. It is known as Kleene normal form theorem for Σ^1_1 relations.

Lemma 1.2.21. *For any Σ^1_1-formula $\phi(n, X)$, there exists a sequence of trees $(T_k)_{k\in\mathbb{N}}$ subsets of $(2\times\omega)^{<\omega}$ such that*

$$\forall n\ \forall X,\ (\phi(n,X) \leftrightarrow \exists f\ \forall k,\ \langle (X(0), f(0)), \dots, (X(k-1), f(k-1)) \rangle \in T_n),$$

where of course we identify X with its characteristic function.

Proof. Using the preceding lemma for a given n, we construct $T_n \subseteq (2\times\omega)^{<\omega}$ as

$$\sigma \in T_n \leftrightarrow \exists f\ \exists m,\ \theta(\sigma[m]).$$

We thus get a sequence of trees with the desired property. □

Proof of the theorem 1.2.19. By the preceding lemma, it is sufficient to prove the following. For any sequence of trees $(T_k)_{k\in\mathbb{N}}$ such that

$$\forall k\ \exists g_k\ g_k \text{ is a branch through } T_k \to \exists (g_k)_{k\in\mathbb{N}}\ \forall k\ g_k \text{ is a branch through } T_k.$$

We shall obtain this as a consequence of Σ^0_1 determinacy.

Consider the following open (Σ^0_1) game. Both players are playing natural numbers. Let k be the first move of player I and g the moves of player II all along a play. The payoff set X contains all the sequences such that g is not a branch through T_k. This set is open since if a sequence is not a branch through a tree, then it has a finite problematic initial segment such that all the sequences in the bouquet of this node are still not branches through the tree. So every point in X is contained in an open neighbourhood in X. Because of our hypothesis, player I cannot have a winning strategy. Invoking open determinacy, we conclude the proof. □

Another theorem for which ATR_0 is the appropriate set of axioms is Suslin's theorem (theorem 1.1.12) which is stated as follows.

Theorem 1.2.22 (ATR_0)**.** *If A_1 and A_0 are codes for analytic sets (of the Cantor space $2^{\mathbb{N}}$) such that $\forall X\ (X \in A_1 \leftrightarrow X \notin A_0)$, then there exists a code for a Borel set B (of the Cantor space) such that $\forall X\ (X \in A_1 \leftrightarrow X \in B)$. Conversely, given any code for a Borel set B there exist analytic codes A_1 and A_0 with these properties.*

Note that "$X \in A_1$" and "$X \in B$" are some Σ^1_1 formulae, the latter stating the existence of a so-called "evaluation map" verifying an arithmetical condition (which is consistent with our analogy between the hierarchies of formulae of logic and definable sets of descriptive set theory).

For the sake of completeness let us state a last theory of L_2 of intermediate strength between RCA_0 and ACA_0.

Definition 1.2.23 (WKL_0)**.** *The L_2-theory of* WKL_0 *consists of the axiom of* RCA_0 *plus the* weak König's lemma*, that is the statement*

$$\forall T \subseteq 2^{\mathbb{N}} \quad T \textit{ infinite} \rightarrow \exists f \;\; f \textit{ is a branch through } T.$$

This lemma is sufficient and necessary to prove that every continuous real-valued function defined on a compact interval $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable. Note that we characterise a compact (complete metric) space by providing for each $j < \omega$, a finite collection of centers for open balls of radius 2^{-j} covering the space. It is also equivalent to several basic theorems of mathematical logic such as Gödel's completeness (its standard form) and compactness theorem.

The formal subsystems of second-order arithmetic RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0 are called the big five. Most of the reverse mathematics results for theorems of ordinary mathematics can indeed be successfully implemented inside one of these subsystems. In addition to that, they turn out to correspond to several philosophical programmes in the foundation of mathematics. Again, we invite the interested reader to consult the book of Simpson [48] for further development around reverse mathematics. Along the way, we also reference the introductory book of Stillwell [50] on the subject "Reverse Mathematics: Proofs from the inside out", as well as the more advanced article of Shore [47], "Reverse Mathematics: The Playground of Logic", which treats of various subsystems of second-order arithmetic lying outside of the big fives, as well as the relation of reverse mathematics with other fields of logic such as computability theory, with an eye on our subject: determinacy of infinite games.

1.3 A set-theoretic interpretation of ATR_0

Despite the reverse mathematics exposition of the last chapter being expressed in the language of L_2, we could wonder if this two-sorted language is necessary to inspect and classify theorems according to their logical strength and find the appropriate axioms to prove them. Even if second-order arithmetic seems to be particularly well suited to such a meticulous analysis, precisely because it takes into account the difference between two kinds of objects, the question remains whether or not the big five are interpretable as some sub-theory of ZFC.

The possibility of a back-and-forth between the two theories would be very fruitful in terms of results for both sides. Moreover, the coding of objects in second-order arithmetic is often quite tedious. When the axioms we use are strong enough like in some of the proofs of the preceding section, our proofs are indeed closer to those of set theory and hence often more readable.

Taking the example of ATR_0, we will see that such a translation is possible! Because of the unbounded capacity of ZFC to create new sets, way bigger than the countable ones, we need to find way weaker, and eventually restrictive axioms for our set interpretation. Therefore we first set up a weaker base theory in L_{Set}, nevertheless suitable for the construction of a reasonable part of classical set theory. All the content of the present section is mostly developed in [48, VII.3]. The very crucial tools presented here, suitable trees coding sets, were introduced by Jäger and Simpson. Similar results can also be found in [2].

Definition 1.3.1 ($\mathsf{B}_{\mathrm{Set}}$)**.** *The L_{Set}-theory $\mathsf{B}_{\mathrm{Set}}$ (*Base set theory*) is axiomatized by the axioms*

Equality:	*The relation $=$ is reflexive, symmetric, transitive, and $\in$ is well defined under this equivalence relation,*
Extensionality:	*If u and v have the same elements, then $u = v$,*
Pair:	*The pair of u and v, $\{u, v\}$, is a set,*
Union:	*The union $\bigcup u$ is a set,*
Infinity:	*There exists an inductive set,*
Δ_0 separation:	*The collection $\{x \in u : \phi(x)\}$, of the sets in u satisfying ϕ is a set for ϕ, a Δ_0 formula,*

where Δ_0 formulae are the one-sorted analogous of Δ^0_0 formulae, with quantifiers bounded by the use of "$\in$".

Notice that here we want to use "=" as a symbol of the language L_{Set} because the equality is not defined between sets in second-order arithmetic, so if we want to interpret back a given set theory in second-order arithmetic, we have to show that EQUALITY is true. Abusing of notations, however write L_{Set} instead of $L_{\text{Set}} \cup \{=\}$ for the language $\{\in, =\}$ during this section.

As it is shown in [48, VII.3], we can indeed implement a reasonable part of the set-theoretic notions inside B_{Set}, like (countable) ordinals. We can therefore set up the following set-theoretic interpretation of second-order arithmetical, expressed in terms of models.

Definition 1.3.2 (Interpretation of L_2 from L_{Set})**.** *To any model $A = (|A|, \in_A)$ of B_{Set} we can canonically associate an L_2 structure*

$$A^2 = M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M).$$

Namely

$$|M| = \omega_A = \{a \in |A| \mid A \models a \textit{ is a finite ordinal}\},$$
$$\mathcal{S}_M = \{b_A : A \models b \subseteq \omega_A \wedge b \in |A|\} \textit{ where } b_A = \{a \in |A| \mid A \models a \in b\},$$

and $+_M, \cdot_M, 0_M, 1_M, <_M$ are defined the standard way.

Of course, A^2 does not preserve, in general, all the information of A since the latter may contain uncountable sets. However, we can extract good properties for A^2 from A.

Lemma 1.3.3. *If A is a model of B_{Set}, A^2 is a model of* ACA_0.

Proof. The basic axioms are easily verifiable. Let us prove the scheme of arithmetical comprehension. Let $\phi(n)$ be an arithmetical formula. We can naturally translate this L_2 formula into one of L_{Set}, $\phi'(u)$. To do this, the symbols $+$, $\cdot$, 0, 1, $=$, $\in$ are interpreted following their standard construction in set theory. The number variables are interpreted as ranging over elements of ω, making of $\phi'(u)$ a Δ_0 formula. Then

$$A \models \exists u \quad u = \{n \in \omega \mid \phi'(u)\},$$

by Δ_0 SEPARATION. Thus, it follows from the definition that

$$A^2 \models n \in b_A \leftrightarrow \phi(n).$$

Thus, the induction scheme follows from arithmetical comprehension together with the set induction

$$\forall X(\subseteq \omega)\ [(0 \in X \land \forall n\ (n \in X \to n+1 \in X)) \to \forall n\ (n \in X)],$$

that we can deduce from ω being defined as well-founded in B_{Set} (see [48, VII.3]). □

To go the converse way we need the following concept.

Definition 1.3.4 (Suitable tree). *We define a* suitable tree *by a well-founded non-empty tree $T \subseteq \mathbb{N}^{\mathbb{N}}$, i.e. T has no path, i.e.*

$$\neg(\exists f \in \mathbb{N}^{\mathbb{N}}\ \forall m\ f[m] \in T).$$

Notice that this is a Π^1_1 condition.

These trees are deemed to encode sets by keeping track of the relation "$a \in b$" with the relation "a is a child of b". To better understand the meaning of the equality and the membership relation between sets coded by suitable trees we introduce the following tools.

Definition 1.3.5 (Regular relation and collapsing function). *A relation r is said to be* regular *if*

$$\forall u\ (u \neq \emptyset \implies \exists x \in u\ \forall y \in u\ ((y,x) \notin r)).$$

A collapsing function for a relation r is a function whose domain is the field of r with the property

$$\forall x \in \mathrm{field}(r)\ (f(x) = f(\{y : (y,x) \in r\})).$$

We already know the following regular relation.

Definition 1.3.6 (Kleene/Brouwer ordering). *Given a tree $T \subseteq \mathbb{N}^{\mathbb{N}}$, we define the* Kleene/Brouwer linear ordering $\mathrm{KB}(T) \subseteq T \times T$ *as the set of couples (σ, τ) such that*

$$\begin{cases} \sigma \supseteq \tau & \text{if the sequences are compatible,} \\ \exists j < \min(|\sigma|, |\tau|)\ [\sigma(j) < \tau(j) \land \forall i < j(\sigma(i) = \tau(i))] & \text{otherwise.} \end{cases}$$

If $T = \mathbb{N}^{<\mathbb{N}}$, then we get a dense linear ordering with no right point and with the empty sequence as a left point.

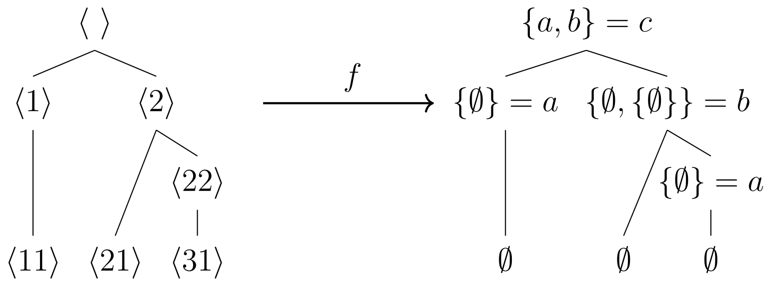

Figure 1.6: Interpretation of suitable trees.

Inside set theory (where we can state a definition similar to 1.3.4), if we have such a collapsing function of the Kleene/Brouwer ordering of a suitable tree. The range of this function will be the standard interpretation of the set that the tree is deemed to code. Given two suitable trees S and T given collapsing function for them c_S and c_T (provided that they exist), we then want to say to $S =^* T$ iff $c_S(S) = c_T(T)$ and $S \in^* T$ iff $c_S(S) \in c_T(T)$.

Figure 1.6 is an example of suitable tree coding the set "$c = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$", translated by the use of its collapsing function, f. Figure 1.7 is an illustration of an equivalent tree under $=^*$ and of the relation $\in^*$, showing that $\{\emptyset, \{\emptyset\}\} = b \in c$. We can interpret $=^*$ as the equivalence relation induced by the collapsing function. As for the relation $\in^*$, we must notice that any subtree $T_{\langle n \rangle}$, which is again a suitable tree is coding the elements coded by T.

It turns out that we can define such an equivalence relation on trees in second-order arithmetic. We define $S \oplus T$ to be the suitable tree consisting of $\langle\rangle$ plus all $\langle 0 \rangle^\frown \sigma$ and $\langle 1 \rangle^\frown \tau$ for $\sigma \in S$ and $\tau \in T$. This equivalence relation in set theory was defined as

$$X = \{(\sigma, \tau) : \sigma,\ \tau \in S \oplus T \wedge c_{S\oplus T}(\sigma) = c_{S\oplus T}(\tau)\}.$$

We can also describe the desired conditions of X inside L_2. We resume them by

$$\mathrm{Iso}(X, S \oplus T).$$

We refer the reader to [48, VII.3] for the detailed (arithmetical) conditions, rather technical and not necessary for the present study.

Definition 1.3.7 ($=^*$ and $\in^*$ in L_2)**.** *Given S and T suitable trees, we define the Σ^1_1 conditions*

$$S =^* T \leftrightarrow \exists X\ \mathrm{Iso}(X, S \oplus T) \wedge ((\langle 0 \rangle, \langle 1 \rangle) \in X)$$

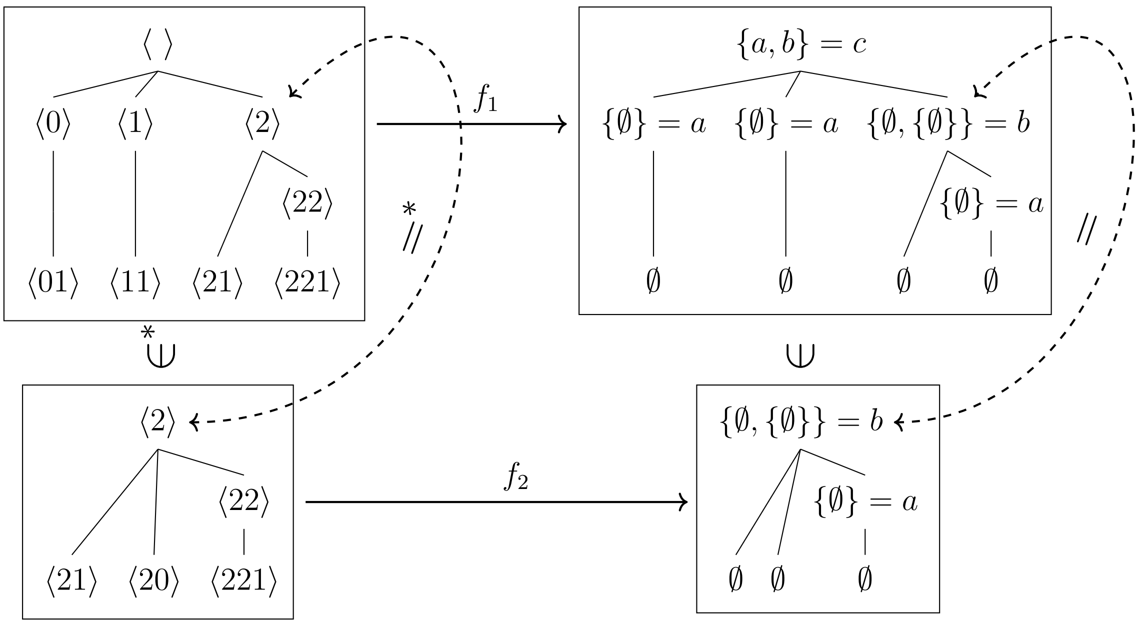

Figure 1.7: Illustration of relations between suitable trees.

and

$$S \in^* T \leftrightarrow \exists X \ \mathrm{Iso}(X, S \oplus T) \wedge \exists n \ ((\langle 0 \rangle, \langle 1, n \rangle) \in X).$$

In ATR_0, we can show that such equivalence relations always exist and are unique for suitable trees.

Definition 1.3.8 (Interpretation of L_{Set} from L_2)**.** *To any L_2-structure*

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

we associate an L_{Set}-structure as follows. Put

$$\mathcal{T}_M = \{T \in \mathcal{S}_M \mid M \models T \text{ is a suitable tree}\},$$

for $T \in \mathcal{T}_M$ put

$$[T] = \{T' \in \mathcal{T}_M \mid M \models T =^* T'\}$$

and define

$$|A| = \{[T] : T \in \mathcal{T}_M\}.$$

For $T_1, T_2 \in \mathcal{T}_M$ define $[T_1] \in_A [T_2]$ iff $M \models T_1 \in^* T_2$. Thus we define

$$M_{\text{Set}} = A = (|A|, \in_A).$$

To ensure the construction to well behave, we usually work inside a model of ATR_0. Similarly to the preceding proposition, we can get basic properties for M_{Set} inside weaker settings. To do this, we also need a translation of formulae, this time from L_{Set} to L_2.

Definition 1.3.9. *Translation of formulae from L_{Set} to L_2 Given an L_{Set} formula ϕ, we define an L_2 formula $|\phi|$ in the following way, with V_i, $i < \omega$ are intended to denote suitable trees and are linked to the set-theoretic variables v_i of ϕ:*

1. *$|v_i = v_j|$ is $V_i =^* V_j$;*
2. *$|v_i \in v_j|$ is $V_i \in^* V_j$;*
3. *$|\neg\phi|$ is $\neg|\phi|$; $|\phi \wedge \psi|$ is $|\phi| \wedge |\psi|$;*
4. *$|\forall v_i\ \phi|$ is $\forall V_i$ (V_i suitable tree $\rightarrow |\phi|$);*
5. *$|\exists v_i\ \phi|$ is $\exists V_i$ (V_i suitable tree $\wedge |\phi|$).*

Then, v_i if free in ϕ iff V_i is free in $|\phi|$.

Such results can be found in [2] for instance. Sometimes, it is also useful to consider an alternative definition of Δ_0 SEPARATION, given by the following folklore result, the proof of which can be found in [23, Chapt. 13].

Theorem 1.3.10 (ZF)**.** *There exists operations $G_1, \dots, G_{10}$ such that if $\phi(u_1, \dots, u_n)$ is a Δ_0 formula of L_{Set}, then there is a composition of $G_1, \dots, G_{10}$ such that for all $v_1, \dots, v_n$,*

$$G(v_1, \dots, v_n) = \{(u_1, \dots, u_n) : u_1 \in v_1, \dots, u_n \in v_n \text{ and } \phi(u_1, \dots u_n)\}.$$

Lemma 1.3.11. ACA_0 *proves $|$INFINITY$|$.*

Proof. Given two natural numbers n and m of M we define

$$\begin{aligned} &n\mathrm{E}m \leftrightarrow n = m_i \text{ where} \\ &m = 2^{m_1} + 2^{m_2} + \dots + 2^{m_j}, \qquad m_1 > m_2 > \dots > m_j. \end{aligned}$$

We define V^0 as the suitable tree consisting of $\langle\rangle$ plus all $\langle n_0, \dots n_k\rangle$ such that $n_{i+1}\mathrm{E}n_i$ for all $i < k$. We claim that $M_{\text{Set}} \models [V^0](=: v_0)$ is inductive. Let v_1 and v_2 be sets in v_0 and countable trees V_1 and V_2 representing them. WLOG we can consider $V_1 = V^0_{\langle n\rangle}$ and $V_2 = V^0_{\langle m\rangle}$ for some natural numbers n, m. Then $v_1 \cup \{v_2\}$ is represented by the tree

V_3 consisting of $\langle\rangle$ plus all the finite sequences $\sigma \in V_1$ and $m^\frown\tau$ with $\tau \in V_2$. Suppose m does not appear in the binary expansion of n then

$$V_3 =^* V^0_{\langle n+2^m\rangle} \to V_3 \in^* V^0.$$

Otherwise it means that $V_2 \in^* V_1$ and so $V_3 =^* V_1$ which by hypothesis satisfies $V^1 \in^* V^0$. Thus, we have proved that $M_{\mathrm{Set}} \models v_1 \cup \{v_2\} \in [V^0]$. □

Nevertheless, since we don't have the scheme of transfinite recursion, some suitable trees in M may not have such an equivalence relation and then we could lose some information present in M by only considering its set-theoretic counterpart.

In order to get better mutual interpretability results, we need to improve the strength of the respective theories. We will now define the set-theoretic equivalent of ATR_0.

Definition 1.3.12 ($\mathsf{ATR}_0^{\mathrm{Set}}$). *The L_{Set}-theory axiomatized by the axioms of $\mathsf{B}_{\mathrm{Set}}$ plus*

Regularity	*If u is nonempty, then it has an $\in$-minimal element,*
Countability	*There exists a transitive and countable set v such that $u \subseteq v$,*
Beta	*If r is a regular relation, its collapsing function is a set,*

is called $\mathsf{ATR}_0^{\mathrm{Set}}$.

Not only this theory gives good properties to the representation of sets by suitable trees by giving a concrete interpretation to $=^*$ and $\in^*$ thanks to the existence of the collapsing function, but it also implies that any set can be coded as a suitable tree.

Proposition 1.3.13 ($\mathsf{ATR}_0^{\mathrm{Set}}$). *Given a set u, there exists a suitable tree T such that, given its collapsing function c_T,*

$$c_T(\langle\ \rangle) = c_T(\{\langle n\rangle : \langle n\rangle \in T\}) = u.$$

Proof. By Countability, consider the countable transitive set v containing u and its injection $i : v \to \omega$. We define $T \subseteq \mathbb{N}^{\mathbb{N}}$, containing $\langle\rangle$ and all sequences $\langle i(v_0), \ldots, i(v_k)\rangle$ such that $v_0 \in u$ and $v_{i+1} \in v_i$ for all $i < k$. By Regularity, T is suitable and by Beta it has a collapsing function c_T, moreover, we can suppose the lemma is true for the elements of u since otherwise it would contradict Regularity. This way, denoting by T_{v_0} the suitable tree representing $v_0 \in u$,

$$\begin{aligned} c_T(\langle\ \rangle) &= c_T(\{\langle n\rangle : \langle n\rangle \in T\}) \\ &= \{c_{T_{v_0}}(T_{v_0}) : v_0 \in u\} = u. \end{aligned}$$

□

It turns out that both theories ATR_0 and $\mathsf{ATR}_0^{\mathrm{Set}}$ are equivalent in terms of provability, which is exposed in the following result of [48, VII.3].

Definition 1.3.14. *Let T_0 be any theory in the language L_2 containing ATR_0 (each axiom of ATR_0 is a theorem of T_0). We define*

$$T_0^{\mathrm{Set}} = \mathsf{ATR}_0^{\mathrm{Set}} + T_0,$$

i.e. T_0^{Set} is that theory in the language L_{Set} whose axioms are those of $\mathsf{ATR}_0^{\mathrm{Set}}$ plus the natural translation into L_{Set} of those of T_0.

Theorem 1.3.15. *Let T_0 be any theory in the language L_2 containing ATR_0. Let M be a model of T_0 and A a model of T_0^{Set}, then*

1. *M_{Set} is a model of T_0^{Set} and A^2 a model of T_0;*
2. *$(M_{\mathrm{Set}})^2 = M$ and $(A^2)_{\mathrm{Set}} = A$, up to canonical isomorphisms.*

Moreover, the hierarchies of logical complexity for formulae are well preserved under this translation.

Theorem 1.3.16 ($\mathsf{ATR}_0^{\mathrm{Set}}$)**.** *Assume $0 \leq k < \omega$.*

1. *If ϕ is a Σ_k formula of L_{Set}, then $|\phi|$ is equivalent to a Σ^1_{k+1} formula of L_2.*
2. *If ϕ is a Σ^1_{k+2} formula of L_2, then it is equivalent to a Σ_{k+1} formula of L_{Set}.*

Finally, we point out the fact that there is a correspondence between the β-models of ATR_0 and transitive models of $\mathsf{ATR}_0^{\mathrm{Set}}$. A model $A = (|A|, \in_A)$ of set theory is said to be transitive if $|A|$ is a transitive set, i.e.

$$\forall x, y \; x \in |A| \text{ et } y \in x \to y \in |A|,$$

and $\in_A = \in \restriction |A|$.

First, since being a suitable tree is a Π^1_1 condition, the suitable trees of a β-model are the ones in the ground model so a subtree of a suitable tree is still suitable and its set-theoretic translation is transitive.

Second, by the usual Kleene normal form argument, suppose that the translation of a transitive model is not a β-model. This means that some relation becomes regular in the model because the infinite descending sequence is no more an element of the model. By Beta, we could consider the standard representation α of this relation. By transitivity, following an infinite descending sequence in the ground model gives us a standard representation for it

$$\alpha \ni \alpha_1 \ni \alpha_2 \ni \cdots \ni \alpha_n \ni \cdots$$

a contradiction to Regularity.

Chapter 2

Set Theory Tools and their Consequences on Subsystems of Second Order Arithmetic

2.1 The Constructible Universe

Back in the context of ZFC, we introduced the interest of studying AD and local versions of determinacy (i.e. for payoff set of a given complexity), by the fact that, despite ZFC being very powerful, lots of mathematical questions remain unsolved. Actually, it is a consequence of Gödel's incompleteness theorem that in every recursively enumerable theory able to prove a sufficient collection of theorems about arithmetic, there are statements that can neither be proved nor disproved. In particular, there are sentences ϕ of L_{Set} and models M_1 and M_2 of ZFC such that ϕ is true in M_1 but false in M_2. An example of such a sentence is the continuum hypothesis, which claims that there is no intermediate cardinality between the cardinality of the natural numbers $\mathbb{N}$ and the real numbers $\mathbb{R}$. In other words, assuming CHOICE, $|\mathbb{R}| = \aleph_1$. Constructible sets were introduced by Gödel to prove the consistency of the axiom of choice and the continuum hypothesis (even the generalized one).

Let $(M, \dots)$ be a structure for some language and $A \subseteq M$ a set of parameters. We say that a set $X \subseteq M$ is A-definable if there is a formula ϕ of the language and some $a_1, \dots, a_n \in A$ such that

$$X = \{x \in M \mid (M, \dots) \models \phi(x, a_1, \dots, a_n)\}.$$

We also say that ϕ defines X. Note that in the context of second-order arithmetic, we coded uncountable definable sets by ϕ. Moreover, let

$$\text{def}[A] = \{X \subseteq A \mid X \text{ is } A\text{-definable over } (M, \dots)\}.$$

Definition 2.1.1 (Constructible universe)**.** *The* constructible hierarchy *is defined by transfinite induction*

1. $L_0 = \emptyset$, $L_{\alpha+1} = \text{def}[L_\alpha]$,
2. $L_\gamma = \bigcup_{\beta<\gamma} L_\beta$ *if* γ *is a limit ordinal, and*
3. $L = \bigcup_{\alpha \in Ord} L_\alpha$,

where Ord *denotes the class of all the ordinal numbers.*

Furthermore, L is the smallest class which is a transitive model of ZF and contains Ord. We call CONSTRUCTIBILITY the axiom whereby every set of the universe is constructible, i.e.

$$\forall x \; x \in \mathrm{L}$$

and we often abbreviate it by "$V = L$" (where V is deemed to denote the universe). We can also take any transitive set X as L_0 and we then write $L(X)$ for the class of sets construction from X. It is worth noticing that $L(X)$ can be defined in $\mathsf{ATR}_0^{\mathrm{Set}}$ for any $X \subseteq \omega$ by coding the defining formulae by natural numbers, which process is called "Gödel numbering". This way, in second-order arithmetic (beginning from ATR_0) we can also state CONSTRUCTIBILITY as

$$\exists X \; \forall Y \; (Y \in L(X)).$$

Among others, further developments about constructible sets can be found in the books of Barwise, Devlin, Jech, Jensen, Martin and Simpson [3, 10, 23, 24, 35, 48] as well as in the book of Devlin adapted for undergraduates, "The Joy of Sets" [11] which is also a very good introduction to general set theory. From there, we can extract the following folklore results, for which we introduce the next preliminary notion about cardinal numbers.

Definition 2.1.2 (Regular cardinal)**.** *Let* α *be an ordinal number. A set* $A \subseteq \alpha$ *is* cofinal *in* α *if for each* $\beta < \alpha$ *there is a* γ *in* A *such that* $\beta \leq \gamma$*. In other words, we can build an unbounded map* $f : A \to \alpha$*.*

The cofinality *of* α, $\mathrm{cf}(\alpha)$*, is the least cardinal size of a cofinal subset in* α*.*

A cardinal α *is said to be* regular *if* $\mathrm{cf}(\alpha) = \alpha$*, otherwise we say that* α *is singular.*

Theorem 2.1.3 (ZF)**.** *For every infinite ordinal* α*,*

1. $|\mathrm{L}_\alpha| = |\alpha|$*;*
2. L_α *is a model of* CHOICE*;*
3. L_α *is a model of* B_{Set} *iff* α *is a limit ordinal, except when* $\alpha = \omega$*, where only* INFINITY *doesn't hold in* L_ω*;*
4. *If* $\aleph_0 < \alpha$ *is a regular cardinal, then* L_α *is a model of* ZF^-*;*

Proof. Firstly, observe that $\alpha \in L_{\alpha+1}$ hence $|\alpha| \leq |L_\alpha|$ for all α. When $\alpha = n$ is finite, we also have $L_{n+1} = \mathcal{P}(L_n)$, showing that L_{n+1} is finite and $|L_\omega| = \aleph_0$. We now prove that $|L_\alpha| \leq |\alpha|$ by transfinite induction. If $\alpha + 1$ is a successor ordinal, since there are countably many definable sets from L_α,

$$|L_{\alpha+1}| \leq \max(|\alpha|, \aleph_0) = |\alpha| = |\alpha + 1|.$$

If β is a limit ordinal,

$$|L_\beta| = |\bigcup_{\alpha<\beta} L_\alpha| = \sum_{\alpha<\beta} |\alpha| \leq \sum_{\alpha<\beta} |\beta| \leq |\beta| \cdot |\beta| = \beta.$$

For the second point, we prove CHOICE by exhibiting the definition of a well ordering of L_α, $<_{L_\alpha}$, definable on L_α. We define $<_{L_\alpha}$ by transfinite induction, the limit case being obvious as we define $<_{L_{\alpha+1}}$ as an extension of $<_{L_\alpha}$. Thus we have three cases to compare $x \neq y \in L_\alpha$:

1. If $x, y \in L_\beta$ for some $\beta < \alpha$, $x <_{L_\alpha} y$ exactly when $x <_{L_\beta} y$;
2. If only x satisfy the preceding condition then $x <_{L_\alpha} y$ and respectively when only y does;
3. If x, y are newly defined elements of L_α we order them according to a fixed well-ordering of their respective defining formulae.

The next point is straightforward to check.

Finally, take α, a regular cardinal. The only non-obvious point is to prove SEPARATION, REPLACEMENT by following a similar argument. We will crucially use the following reflection principle. Given $\delta < \alpha$ and a formula $\exists x \phi(x)$ with parameters in L_δ, we have

$$L_\alpha \models \exists x \phi(x) \to \exists \delta < \gamma < \alpha \; L_\gamma \models \exists x \phi(x).$$

Using regularity, we now show that there is a $\beta < \alpha$ such that L_β satisfies the same formulae as L_α, we will call it, an elementary submodel of L_α. Since there are countably many formulae of the form presented above consider B, the set of the γ reflecting such formulae. Since B is countable but α is regular, $\sup(B) = \beta_0 < \alpha$. We then reiterate the process to form a sequence $\delta < \beta_0 < \beta_1 < \cdots < \beta_n < \cdots < \alpha$. Taking then $\beta = \sup_{n<\omega} \beta_n$, which is again by regularity strictly less than α, L_β is what we call the Skolem Hull of L_δ and is the desired elementary submodel of L_α (by applying Tarski-Vaught test for elementary submodels).

Now we prove SEPARATION. Let $\phi(x)$ with a formula with parameters in L_α and $a \in L_\alpha$. We have to show that

$$S_a^\phi = \{x \in a \mid \phi(x)\} \in L_\alpha.$$

Let $\delta < \alpha$ such that L_δ contains a and all the parameters of ϕ. Then L_β the Skolem hull of L_δ such that each

$$L_\alpha \models \exists x \phi(x) \leftrightarrow L_\delta \models \exists x \phi(x),$$

with $\delta < \alpha$. Then it is clear that $S_a^\phi \in L_{\delta+1} \subset L_\alpha$, concluding our proof. □

We can improve the idea of the last result, to a kind of a local version, but first, let us set up the foundational notions we just used.

Definition 2.1.4 (Elementary embeddings)**.** *Let $\mathcal{M} = (M, \in_M)$ and $\mathcal{N} = (N, \in_N)$ be structures of L_{Set} and $j : M \to N$ be an embedding. We say that j is a Σ_k elementary embedding if*

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \leftrightarrow \mathcal{N} \models \phi(j(a_1), \ldots, j(a_n)),$$

for all Σ_k formula of L_{Set} and $a_1, \ldots, a_n \in M$. We say that j is elementary of the preceding holds for any formula of L_{Set} and write $\mathcal{M} \cong \mathcal{N}$ if j is an elementary bijective embedding.

When j is the inclusion map, we write $\mathcal{M} \preceq_k \mathcal{N}$ to say that $\mathcal{M}$ is an Σ_k elementary substructure of $\mathcal{N}$ or that $\mathcal{N}$ is a Σ_k elementary extension of $\mathcal{M}$.

A useful characterisation of the elementary submodel is the Tarski-Vaught criterion that we already used in the proof of theorem 2.1.3. For its proof, we refer to any introduction on model theory, like [33].

Theorem 2.1.5 (Tarski-Vaught criterion)**.** *Suppose that $\mathcal{M}$is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$is an elementary substructure if and only if, for any formula $\phi(x)$ with parameters in $\mathcal{M}$*

$$\mathcal{N} \models \exists x \phi(x) \leftrightarrow \mathcal{M} \models \exists x \phi(x).$$

Another central result is the condensation lemma. We omit its proof which can be found in any good book about constructibility like [10].

Theorem 2.1.6 (Condensation lemma)**.** *Let α be a limit ordinal. If*

$$X \prec_1 L_\alpha,$$

then there are unique π and β such that $\beta \leq \alpha$ and:

1. $\pi\colon (X, \in) \cong (L_\beta, \in)$;
2. *if $Y \subseteq X$ is transitive, then* $\pi_{|Y} = \text{id}_{|Y}$;

3. $\pi(x) \leq_{L_\alpha} x$ *for all* $x \in X$,

with $\leq_{L_\alpha}$ *defined as in the proof of point* 3 *of theorem 2.1.3.*

We often call π, the Mostowski collapsing function. For the following important characterisation of COMPREHENSION in constructible universes, we need some central results that have some flavour of Gödel's second incompleteness theorem, but in a more syntactical way. The proof we present is a straightforward generalisation of the sketch of the proof exposed in [23] about this theorem of Tarski.

Theorem 2.1.7 (Tarski's undefinability of truth)**.** *Let $\mathcal{L}$ be a language containing L_{Set} of cardinality α and $\sharp\sigma$ be a Gödel numbering of the sentences $\mathcal{L}$ over $|\omega \cdot \alpha|$. Then there is no* truth definition. *That is, there is no predicate $T(x)$ such that*

1. $\forall x \ (T(x) \to x \in |\omega \cdot \alpha|)$;
2. *If σ is a $\mathcal{L}$-sentence, then* $\sigma \leftrightarrow T(\sharp\sigma)$.

Proof. Let us assume that a definition of truth $T(x)$ exists. Let

$$\phi_0, \phi_1, \ldots \phi_\beta, \ldots$$

be an enumeration of all $\mathcal{L}$ formulae with one free variable. Let $\psi(x)$ be the formula

$$x \in |\omega \cdot \alpha| \wedge \neg T(\sharp(\phi_x(x))).$$

There is an ordinal number $\gamma < |\omega \cdot \alpha|$ such that ψ is ϕ_γ. Let σ be the sentence $\psi(\gamma)$. Then we have

$$\sigma \leftrightarrow \psi(\gamma) \leftrightarrow \neg T(\sharp(\phi_\gamma(\gamma))) \leftrightarrow \neg T(\sharp\sigma),$$

a contradiction. □

Theorem 2.1.8. *Let α be an ordinal and n, a natural number, the following conditions are equivalent:*

1. $L_\alpha \models \Sigma_n$ *SEPARATION*;
2. $\forall \beta < \alpha \ \exists \gamma \ (\beta < \gamma < \alpha \wedge L_\gamma \preceq_n L_\alpha)$.

In particular, if they hold, then $L_\alpha \models \Sigma_n$ REPLACEMENT.

Proof. We first prove that 1 implies 2. Let $\omega < \beta < \alpha$. Define recursively

$$H_1 = \Sigma_n(L_\beta) = \{x \in L_\alpha \mid \exists \phi \in \Sigma_n \ \exists y \in L_\beta \ (L_\alpha \models \phi(x,y) \wedge \forall y <_L x \ L_\alpha \not\models \phi(x,y))\},$$
$$H_2 = \Sigma_n(X_1), \ldots, H_{k+1} = \Sigma_n(H_k), \ldots;$$
$$H = \bigcup_{k<\omega} H_k.$$

We claim that H is transitive. Indeed, since there are countably many formulae, for all k there is a counting $j_k : \beta \mapsto y$ for each $y \in H_k$, which belong to H_{k+1}. Then the value of this counting for any $\delta < \beta$ is also in H_{k+1} and so does every element of y, proving our claim.

We now show by induction that $H \preceq_n L_\alpha$. Suppose $H \preceq_i L_\alpha$ for $i < n$, we want to show $H \preceq_{i+1} L_\alpha$. Let ϕ be a Π_i formula with parameters from H_k, for some $k < \omega$ such that $L_\alpha \models \exists x \phi(x)$ (we use Tarski-Vaught test for elementary embeddings). Then there exists an $x \in H_{k+1}$ such that $H \preceq_i L_\alpha \models \phi(x)$, proving our claim. Thus, by theorem 2.1.6, $H = L_\delta$ for some $\delta \leq \alpha$.

Finally, we show that $\delta < \alpha$. Indeed consider the following set coding H,

$$C = \{(\phi, \bar{p}_\phi, \psi, \bar{p}_\psi) \mid \bar{p}_\phi, \bar{p}_\psi \in L_\beta \cup H \ \wedge \exists! x \ \psi(x, \bar{p}_\phi) \wedge \exists! y \ \psi(y, \bar{p}_\psi) \wedge \forall x, y \ ((\phi(x, \bar{p}_\phi) \wedge \psi(y, \bar{p}_\psi)) \rightarrow x \in y)\},$$

where any $p \in H$ is coded by its definition along the construction of the H_k's, a finite sequence of formulae, which are all Π_{n-1}. Notice that in virtue of Σ_n SEPARATION, we have $C \in L_\alpha$ and we can embed C in $\beta^{<\omega}$ and thus in β. Then, supposing $H = L_\alpha$, we have a truth definition in the language $L_{\text{Set}} \cup \beta$ for the structure L_α, a contradiction to theorem 2.1.7, concluding the proof.

The proof that 2 implies 1 and that $L_\alpha \models \Sigma_n$ REPLACEMENT is the same idea as the one developed in the proof of point 4 of theorem 2.1.3.

□

2.2 Admissible Sets

During our meticulous analysis of the complexity required to prove determinacy on the edge of second-order arithmetic in the next chapter, we will need a more flexible and sensitive hierarchy of set theories, weaker than ZFC. Let us begin with the axioms of admissible sets, whose theory is extensively developed in [3].

Definition 2.2.1 (KP)**.** *The* Kripke-Platek L_{Set}*-theory,* KP*, is axiomatized by the basic axioms*

EMPTYSET: $\exists x \ \forall y \ y \notin x$, *PAIR,*
EXTENSIONALITY, *UNION,*

which we already defined in the preceding chapter (see definition 1.0.1). We also have an adapted version of the regularity axiom, the scheme of FOUNDATION:

$$\exists x \ \phi(x) \rightarrow \exists x \ (\phi(x) \wedge \forall y \in x \ \neg\phi(y)),$$

plus the schemes of Δ_0 SEPARATION *(see definition 1.3.1) and* Δ_0 COLLECTION*, the latter being*

$$\forall x \in u \ \exists y \ \phi(x, y) \to \exists v \ \forall x \in u \ \exists y \in v \ \phi(x, y),$$

for all Δ_0 *formulae in which* v *does not occur free.*

Notice that since we don't require INFINITY to hold, we add an "empty set axiom" asserting the existence of at least one element (for the sake of definiteness, the existence of the set with no elements). Instead, we could suppose that any structure, that is any interpretation of a formal language, is non-empty.

Definition 2.2.2 (Admissible set)**.** *An* admissible *set is a model of* KP

$$(A, \in_A),$$

where $\in_A$ *is the restriction to* A *of the membership relation* $\in$ *and* A *is a transitive set.*

It can be proved that any admissible set actually satisfies Σ_1 COLLECTION and Δ_1 SEPARATION. We also notice the difference between REPLACEMENT and COLLECTION. Using the two last consequence of KP, it can be showed that Σ_1 replacement is also a theorem of KP.

Remark 2.2.3. There are common extensions of $\mathsf{ATR}_0^{\mathrm{Set}}$ and KP. Notably, BETA is not provable in KP without Σ_1 SEPARATION and it follows from the translation of theorem 1.3.16 that $|\Sigma_1$ COLLECTION$|$ would imply Σ^1_2-AC_0 (see definition 1.2.18), which seems to be the weakest formal system to prove $|\mathsf{KP}|$ (here $|T|$ for an L_{Set}-theory containing $\mathsf{ATR}_0^{\mathrm{Set}}$, except possibly the axiom of countability, denotes the translation of this theory in second-order arithmetic, after possibly adding the axiom of countability). More naturally, it can be shown that Σ^1_2-AC_0 is equivalent to Δ^1_2-CA_0 (see definition 2.3.2), so that,

$$|\mathsf{KP} + \text{INFINITY} + \text{BETA}| \approx |\mathsf{ATR}_0^{\mathrm{Set}} + \Delta_1 \text{ SEPARATION}|.$$

The later analysis is based on [48, VII.3]. In the frame of our study, this is to be put in relationship with the respective results of Steel that Σ^0_1-Det is equivalent to ATR_0 over ACA_0 and to the existence of a well-founded model of KP + INFINITY.

Admissible sets naturally arise in the constructible hierarchy we introduced in the last section. In particular, there are infinitely many ordinals $\alpha < \omega_1$ with L_α being admissible. This can be shown by using Downward Lowenheim-Skolem on any substructure $L_{\beta+1} \subsetneq L_{\omega_1}$ to get a countable model of KP, $\mathcal{M}$ and then use the Mostowski collapsing function to get a countable initial segment of the constructible hierarchy

$(L_\alpha, \in) \cong (\mathcal{M}, \in)$, with $L_\beta \in L_\alpha$. Moreover L_α is the smallest model of KP with the class of ordinals of order type α.

We then have a generalized notion of admissible sets which gives us a convenient hierarchy of set theories.

Definition 2.2.4 (n-admissibility)**.** *For any $1 \leq n < \omega$ we say that a set A is n-admissible if*

1. *A is admissible,*
2. *$(A, \in_A)$ is a model of Σ_{n-1} Separation and Δ_{n-1} Collection.*

We say that an ordinal α is n-admissible if L_α is n-admissible.

Let us present some equivalent characterisations of n-admissibility in the constructible universe, essentially similar to the ideas we developed in theorem 2.1.8. Together with the results of section 1.3, they are essential preliminaries for the theorems presented in [38] about determinacy that we will treat in the next chapter.

Proposition 2.2.5. *Let α be an ordinal, the following assertions on L_α are equivalent:*

1. *It is n-admissible;*
2. *It satisfies Σ_n bounding:*

   $$\forall \delta < \alpha \quad (L_\alpha \models \forall \gamma < \delta \; \exists y \; \phi(\gamma, y)) \rightarrow \exists \lambda < \alpha \; L_\alpha \models (\forall \gamma < \delta \; \exists y \in L_\lambda \; \phi(\gamma, y)),$$

   where ϕ is Π_{n-1} with parameters from L_α;
3. *For any function f with domain some $\delta < \alpha$ which is Σ_n (equivalently Π_{n-1}) over L_α, $f[\gamma] \in L_\alpha$ for every $\gamma < \alpha$.*

Thus we have the classical bounded quantifier elimination rule: For any Π_{n-1} formula ϕ, "$\forall x \in t \; \exists y \; \phi$" is equivalent to a Σ_n formula.

Let us now develop some folklore machinery useful when working inside our present setup.

First, we say that a structure of the language of set theory $\mathcal{M}$ has Σ_n Skolem functions if there is a function h of the structure which associates a witness to all Π_{n-1} formula ψ such that $\mathcal{M} \models \exists x \psi(x)$. In other words, a Σ_n Skolem function of $\mathcal{M}$ is a partial function $h\colon \omega \times M \rightarrow M$, $h \in M$ such that for all set A that is Σ_n definable with some parameters $\bar{p}$, there is an i such that $h(i, \bar{p}) \in A$.

Lemma 2.2.6. *If α is n-admissible, then L_α has a parameterless Σ_{n+1} Skolem function.*

Proof. Consider a Π_n formula $\psi(x, \bar{p})$ with Gödel number i. We define

$$h(i, \bar{p}) = x \leftrightarrow \psi(x, \bar{p}) \wedge \forall(x' <_{L_\alpha} x)\ \neg\psi(x, \bar{p}).$$

The first conjunct is thus Π_n while the second is Σ_n by proposition 2.2.5. □

Lemma 2.2.7. *If L_α is n-admissible, then it satisfies Δ_n SEPARATION, that is, for any $u \in L_\alpha$ and Σ_n formulae $\phi(z)$ and $\psi(z)$ such that $L_\alpha \models \forall z\ (\phi(z) \leftrightarrow \neg\psi(z))$, $\{z \in u \mid \phi(z)\} \in L_\alpha$.*

Proof. When $n = 1$, it is a standard fact, as discussed earlier. We now prove our claim by induction for $n \geq 2$. Let $\phi(x, y, z)$ and $\psi(x, y, z)$ be two Σ_{n-2} formulae such that

$$L_\alpha \models \forall z(\exists x\ \forall y\ \phi(x, y, z) \leftrightarrow \neg\exists x\ \forall y\ \psi(x, y, z))$$

and $u \in L_\alpha$. We define on L_α a function with domain u,

$$\begin{aligned} f(z) = x \quad \leftrightarrow \quad & [\forall y\ \phi(x, y, z) \wedge \forall(x' <_{L_\alpha} x)\ \exists y'\ \neg\phi(x, y', z)] \\ & \vee\ [\forall y\ \psi(x, y, z) \wedge \forall(x' <_{L_\alpha} x)\ \exists y'\ \neg\psi(x, y', z)]. \end{aligned}$$

By proposition 2.2.5, the definition of f is equivalent to a Δ_n formula and thus its range is bounded, say, by L_γ. It means that we can find in L_γ enough elements to witness the truth of our defining formula for each $z \in u$. Consequently,

$$\begin{aligned} \{z \in u \mid L_\alpha \models \exists x\ \forall y\ \phi(x, y, z)\} &= \{z \in u \mid L_\alpha \models \exists(x \in L_\gamma)\ \forall y\ \phi(x, y, z)\} \\ &= \{z \in u \mid L_\alpha \models \forall(x \in L_\gamma)\ \exists y\ \neg\psi(x, y, z)\}. \end{aligned}$$

Again using proposition 2.2.5 we deduce that the above set is Δ_{n-1} and thus belongs to L_α, which concludes our proof. □

Lemma 2.2.8. *For any ordinals $\gamma < \beta$, if $L_\gamma \preceq_n L_\beta$ and β is $(n-1)$-admissible, then γ is n-admissible.*

Proof. We proceed by induction on $n \geq 1$, with the convention that 0-admissibility is just transitivity. Suppose towards a contradiction that the lemma fails, it means that there is a Π_{n-1} definable function over L_α such that $f[\gamma]$ is unbounded in L_α for some $\gamma < \alpha$. Then f is obviously bounded in L_β and since it is defined by a Π_{n-1} formulae $\psi(\eta, y)$, by Σ_n elementary embedding, we have

$$L_\beta \models \exists x\ \forall(\eta < \delta)\ \exists(y <_{L_\beta x})\ \psi(\eta, y). \tag{2.1}$$

By using once again proposition 2.2.5 to manipulate quantifiers, the property in 2.1 is equivalent to a Σ_n one, π, in L_β and then, by Σ_n elementary embedding we have $L_\gamma \models \pi$. Since by induction L_γ is $n-1$-admissible, π is still equivalent in L_γ to the property in 2.1, which leads to the desired contradiction and concludes the proof. □

Lemma 2.2.9. *Let n be a natural number and β_n be the smallest ordinal β such that $L_\beta \models \mathsf{ZF}^- +$ "$\mathcal{P}^n(\omega)$ exists", then for all $\gamma < \beta_n$, L_γ has cardinality bounded by $\mathcal{P}^n(\omega)$ in $L_{\gamma+1}$.*

Proof. Once again we proceed by induction, starting with $n = -1$, where our claim is that L_γ is countable in $L_{\gamma+1}$ and $\beta_{-1} = \omega$ (where we remove INFINITY from ZF^-). This case is obviously true. Note also that $\mathcal{P}^0(\omega) := \omega$ (then we ask for INFINITY to hold).

Let $n \geq 0$. For $\gamma = \beta_{n-1}$, the conclusion is immediate from our inductive hypothesis. Now we proceed by transfinite induction. If $\gamma + 1$ is a successor, the conclusion follows from the countable number of formulae and the bound given by induction on the cardinality of the parameter space, L_γ. If γ is a limit ordinal less than β_n, then it is not k-admissible for some $k < \omega$. It means that there is a Σ_n definable map f over L_γ such that $f[\delta]$ is unbounded in L_γ for some $\delta < \gamma$. We can now define a surjective function from $\mathcal{P}^n(\omega)$ to L_γ by combining the maps $g_\delta \colon \mathcal{P}^n(\omega) \to L_\delta$ and $g_{f(\zeta)} \colon \mathcal{P}^n(\omega) \to L_{f(\zeta)}$ given by the induction hypothesis. □

Lemma 2.2.10. *Let α_n be the first n-admissible ordinal, then every element of L_{α_n} is Σ_{n+1} definable over L_{α_n} without parameters.*

Proof. By lemma 2.2.6 let us choose f, a parameterless Σ_{n+1} Skolem function for L_{α_n}. Let H be the Skolem hull of the empty set under f (as in the proof of theorem 2.1.8). First, H is transitive, since by lemma 2.2.9, there is a Σ_n counting of every $\delta \in H$, which then also belongs to H, as well as the value of every $k < \omega$, which are the elements of δ. Thus by theorem 2.1.6, H is a L_γ for somme $\gamma \leq \alpha_n$. On the other hand, $H \preceq_{n+1} L_{\alpha_n}$ so in case $\gamma < \alpha_n$ lemma 2.2.8 would imply that L_γ is $(n+1)$-admissible, a contradiction to the definition of α_n. Thus, $H = L_{\alpha_n}$ and so every element of L_{α_n} is Σ_{n+1} definable (as the value of f for some $k < \omega$) over L_{α_n} without parameters. □

The following result is a corollary of lemma 2.2.7 and theorem 1.3.16.

Theorem 2.2.11. *Let α_n denote the first n-admissible ordinal, then $\mathcal{P}(\omega) \cap L_{\alpha_n}$ is a β-model of Δ^1_{n+1}-CA_0 for $n \geq 2$.*

This theorem (actually 2.2.7) can be viewed as a generalisation of remark 2.2.3. Indeed it is proved in [48, VII.6] that assuming constructibility, we have a generalized version of the equivalence between choice schemes and comprehension schemes in second-order arithmetic. Namely, assuming $\exists X\ \forall Y\ (Y \in L(X))$, it is provable in ATR_0 that

$$\Sigma^1_{k+3}\text{-}\mathsf{AC}_0 \text{ is equivalent to } \Delta^1_{k+3}\text{-}\mathsf{CA}_0,$$

for any $k \in \omega$.

The last theorem also echoes the analysis in [48, VII.5] according to which the minimal β-models of Δ^1_n-CA_0 are initial segments of the constructible hierarchy. These are the kind of results that we will evoke in the following section.

2.3 Conservation Results from Constructibility Hypotheses

Back to constructibility, we should add that a very desirable property of models $\mathcal{M}$ of set theory is *absoluteness*. A formula is said to be absolute relative to a model $\mathcal{M}$ if it has parameters in the domain of $\mathcal{M}$ and it is true whenever its relativisation to $\mathcal{M}$ is true as well. The constructible universe is what is called an *inner model* of ZF, that is, a transitive class that contains all the ordinals and satisfies the axiom of ZF (and more in our specific case). In particular, Δ_0 formulae are *absolute* for every transitive class M. When introducing the constructible hierarchy, we mentioned that such construction as the one of L was translatable in the realm of second-order arithmetic. It turns out that we can get a more powerful absoluteness result in this specific setting. Thus we state it in the language of second-order arithmetic.

Theorem 2.3.1 (Shoenfield absoluteness in Π^1_1-CA_0, [48], VII.4.14)**.** *The following is provable in Π^1_1-CA_0. Let $X \subseteq \mathbb{N}$ and ϕ be any Σ^1_2 sentence with parameter from $L(X)$. Then ϕ is absolute to $L(X)$.*

The above theorem implies that we can suppose constructibility to prove Σ^1_2 sentences, and actually even more. This is convenient since we have a lot of convenient results to work with when reasoning in L, in particular concerning the axiom of choice. Let us state hereby some of the axiom schemes related to it.

Definition 2.3.2. *Assume $0 \leq k < \omega$.*

1. *Σ^1_k-DC_0 is the L_2-theory whose axioms are those of ACA_0 plus the scheme of Σ^1_k dependent choice:*

$$\forall n\ \forall X\ \exists Y\ \eta(n, X, Y) \rightarrow \exists Z\ \forall n\ \eta(n, (Z)^n, (Z)_n)$$

where $\eta(n, X, Y)$ is a Σ^1_k formula in which Z does not occur. We are using the notation

$$(Z)^n = \{(i, m) : (i, m) \in Z \wedge m < n\}$$

and $(Z)_n$ is like in 1.2.18.

2. *Strong* Σ^1_k-DC_0 *is the* L_2*-theory whose axioms are those of* ACA_0 *plus the scheme of* Σ^1_k *strong dependent choice:*

$$\exists Z\ \forall n\ \forall Y\ (\eta(n,(Z)^n,Y) \to \eta(n,(Z)^n,(Z)_n))$$

where $\eta(n,X,Y)$ *is above.*

3. *If* Γ *is a class of formulae,* Γ-CA_0 *is the subsystem of* Z_2 *which consists of the basic axioms plus the scheme of* Γ *comprehension:*

$$\exists X\ \forall n\ (n \in X \leftrightarrow \psi(n))$$

where ψ *is any* Γ *formula in which* X *does not occur freely.*

Remark 2.3.3. We can show easily that strong Σ^1_k-DC_0 implies Σ^1_k-DC_0.

Non unrelatedly with the statement of theorem 2.2.11, the following result is among other uses the constitutive basis for the assertion that the minimal models Δ^1_n-CA_0 are initial segments of the constructible hierarchy.

Theorem 2.3.4 ([48, VII.5.10]). *Let* M' *be any model of* Π^1_1-CA_0*. Given* $X \in S_{M'}$*, let* M *be the* ω *sub-model of* M' *consisting of all* $Y \in S_{M'}$ *such that* $M' \models Y \in \mathrm{L}(X)$*. Then:*

1. M *is a model of* Π^1_1-CA_0 *and* $V = \mathrm{L}(X)$*.*
2. M *is a* β_2*-sub-model of* M'*. That is, for any* Σ^1_2 *sentence* ϕ *with parameters from* M*,* $M \models \phi$ *iff* $M' \models \phi$*.*
3. *If* M' *is an* ω*-model or* β*-model, then so is* M*.*

Furthermore, for all $k \geq 0$*, we have:*

1. *To any* Σ^1_{k+2} *formula* $\phi(n_1,\ldots,n_i,X_1,\ldots,X_j)$ *with parameters from* M *we can associate a* Σ^1_{k+2} *formula* ϕ' *such that, for all* $n_1,\ldots,n_i \in |M|$ *and* $X_1,\ldots X_j \in S_M$*,*

$$M \models \phi(n_1,\ldots n_i,X_1,\ldots,X_j)$$

if and only if

$$M' \models \phi'(n_1,\ldots n_i,X_1,\ldots,X_j).$$

2. *If* M' *is a model of* Γ^1_{k+2}-CA_0*, so is* M *for* $\Gamma \in \{\Pi,\Delta\}$ *and* $k \in \omega \cup \{\infty\}$*.*

Furthermore, it provides us with a construction that can be employed to build up constructible versions of models of subsystems of second-order arithmetic, which can be used to conclude conservation results. The following technique of proof for setting up a conservation result is very important in the frame of our study. We say that a theory $T_0' \supset T_0$ is Π^1_k conservative over T_0 if any Π^1_k sentence T provable in T_0' is also provable in T_0.

Corollary 2.3.5 ([48, VII.5.11])**.** *Let T_0 be Π^1_∞-CA_0, Π^1_{k+1}-CA_0 or Δ^1_{k+2}-CA_0 for $0 \leq k < \omega$. Then $T_0' = T_0 + \exists X\ V = \mathrm{L}(X)$ is Π^1_4 conservative over T_0.*

Proof. We reason in terms of models. Consider a non provable Π^1_4 sentence $T = \forall X\ \exists Y\ \phi(X, Y)$ in T_0. By Gödel's completeness theorem, take any model M' of $T_0 + \neg T$. Thus for a counterexample X, M' satisfies $\neg\phi(X, Y)$ for any Y. In particular, $\exists X \forall Y \neg\phi(X, Y)$ remains true in M constructed as in the preceding theorem. The model M satisfies this way $T_0' + \neg T$. By the soundness theorem, it follows that T is not provable from T_0'. □

As announced earlier, we can deduce from theorem 2.3.1 conjunctively with some work, that we can use constructible hypothesis to prove any Π^1_4 sentence. Let us now expose some useful results of special interest for us that hold in L.

Theorem 2.3.6 ([48, VII.6.16])**.** *The following is provable in ATR_0. Assume $0 \leq k < \omega$ and $\exists X\ V = \mathrm{L}(X)$. Then:*

1. *Σ^1_{k+3}-AC_0 is equivalent to Δ^1_{k+3}-CA_0.*
2. *Σ^1_{k+3}-DC_0 is equivalent to Δ^1_{k+3}-CA_0 plus Σ^1_{k+3}-IND.*
3. *Strong Σ^1_{k+3}-DC_0 is equivalent to Π^1_{k+3}-CA_0.*
4. *Σ^1_∞-DC_0 is equivalent to Π^1_∞-CA_0.*

Now we can carry on these results as conservation statements.

Corollary 2.3.7 ([48, VII.6.20])**.** *Assume $0 \leq k < \omega$.*

1. *Σ^1_{k+3}-AC_0 is Π^1_4 conservative over Δ^1_{k+3}-CA_0.*
2. *Σ^1_{k+3}-DC_0 is Π^1_4 conservative over Δ^1_{k+3}-CA_0 plus Σ^1_{k+3}-IND*
3. *Strong Σ^1_{k+3}-DC_0 is Π^1_4 conservative over Π^1_{k+3}-CA_0.*

Finally, it is a known fact that $L \models \mathsf{ZFC}$. Our second-order arithmetic version of L corresponds in fact to L_{ω_1}, with ω_1 being the first uncountable ordinal. It is also a regular cardinal and as a matter of fact, we can argue in Z_2 that $L(X)$ is gonna be a model of ZFC^-, essentially due to the results we exposed in the present section. This way as it is presented in the introduction of [38], we can use the same reasoning that we used in the proof of corollary 2.3.5 to show the next theorem.

Theorem 2.3.8. *The theory ZFC^-, even with a definable well ordering of the universe assumed as well, is a Π^1_4 conservative extension of Z_2.*

Chapter 3

Some Logical Bounds around Determinacy on the Edge of Systems of Arithmetic

3.1 The Limit of Determinacy in Second Order Arithmetic

The following result was first proved by Wolfe in 1955.

Theorem 3.1.1 ($\mathsf{ZC}^- + \Sigma_1$-Replacement)**.** *All Σ^0_2 games are determined.*

Lemma 3.1.2. *Let $B \subseteq A \subseteq [T]$ with B being closed. If Anais has no winning strategy in the game $G(T, A)$, then there is a strategy τ for Bruce such that every $x \in [\tau]$ has a finite initial segment p verifying*

$$[T_p] \cap B = \emptyset \qquad \textit{and} \qquad \textit{Anais has still no winning strategy in } G(T_p, A).$$

Proof. Assume $G(T, A)$ is not a win for Anais. Let P be the union of bouquets around the sequences p in T satisfying the two properties announced. Therefore, we want to prove that $G(T, \bar{P})$ is a win for Bruce. Since P is open, by open determinacy (theorem 1.1.6), we would otherwise have that Anais has a winning strategy σ. Note that $\sigma \subseteq P$. Our goal is to prove that σ is also winning in the original game, a contradiction. In this scope, suppose that Bruce plays only moves that are non-losing in the original game $G(T, A)$ (as in the proof of theorem 1.1.6). We are then playing in $G(W, \bar{P})$ where $W \subseteq T$ is called the non-losing quasistrategy of Bruce (for $G(T, A)$).

Consider $p \in \sigma \cap W$, since p is non-losing for Bruce in $G(T_p, A)$ but $p \in \sigma \subset P$ we must have $[T_p] \cap B \neq \emptyset$. Because B is closed and p was arbitrary, any $x \in [\sigma]$ (the limit of his finite initial segments $p \subset x$) is also in $B \subseteq A$, making of σ a winning strategy in $G(W, A)$ and so in $G(T, A)$, a contradiction. □

Proof of theorem 3.1.1. Let T be a pruned tree and $A \subseteq [T]$ a Σ^0_2 set,

$$A = \bigcup_{i<\omega} A_i,$$

with closed sets A_i. Suppose Anais has no winning strategy in $G(T, A)$. We describe informally how Bruce can use the preceding lemma to set up a winning strategy τ. First, apply the lemma for $B = A_0$ and play the given strategy τ_0 until you reach a position p_0 verifying the predicted property of τ_0. Iterate this process ω times. After this point, τ can be arbitrary since by construction $[\tau] \cap A_i = \emptyset$ for all $i < \omega$, thus τ is winning for Bruce and we have proved the theorem. □

Empowerment of the methods of this theorem was done by Davis in 1964 to prove Σ^0_3 determinacy in the same setting. However, we present now a generalisation of this result (from [38]), while the proof in itself, without the reverse mathematics framework is a weakening of the (unpublished) proof of Δ^0_4 determinacy of Martin. It turns out that the proof of Martin cannot be implemented inside second-order arithmetic.

From now on we fix $m \in \omega$, $m \geq 1$. We will show that second-order arithmetic can prove the determinacy of games "between" Π^0_3 and Π^0_4. Precisely we introduce the following hierarchy.

Definition 3.1.3 (Hierarchy of differences (for Π^0_α sets))**.** *In any topological space X, given a natural number n and an ordinal number α, we say that a set A is $(\Pi^0_\alpha)_m$ if there are Π^0_α sets $A_0, A_1, \ldots A_{m-1}, A_m = \emptyset$ such that*

$$x \in A \leftrightarrow \textit{the smallest } i \textit{ such that } x \notin A_i \textit{ is odd.}$$

We say that the sequence $\{A_i : i \leq m\}$ represents A (as an $(\Pi^0_\alpha)_m$ set).

Proposition 3.1.4. *For any $m < 0$, taking A as defined in the preceding definition, we can suppose that for all $0 < i < m$, $A_i \subseteq A_{i-1}$.*

Proof. We prove this by induction on i. Since 0 is even and the smallest natural number, we can suppose that every $A_i \subseteq A_0$. Suppose $1 < i < m$ and $x \in A_i \setminus A_{i-1}$, since by induction hypothesis $A_{i-1} \subseteq A_{i-2} \subseteq \cdots \subseteq A_0$ and we are interested in the smallest j such that $x \notin A_j$ these preceding sets already takes into account whether or not $x \in A$ and we can suppose $A_i \subseteq A_{i-1}$, since otherwise we could replace A_i by $A_i \cap A_{i-1}$. □

The figure 3.1 is an example of a $(\Gamma)_5$ set for a class $\Gamma = \Pi^0_\alpha$ for some α and depicts the idea that we can suppose the sequence to be nested backwards.

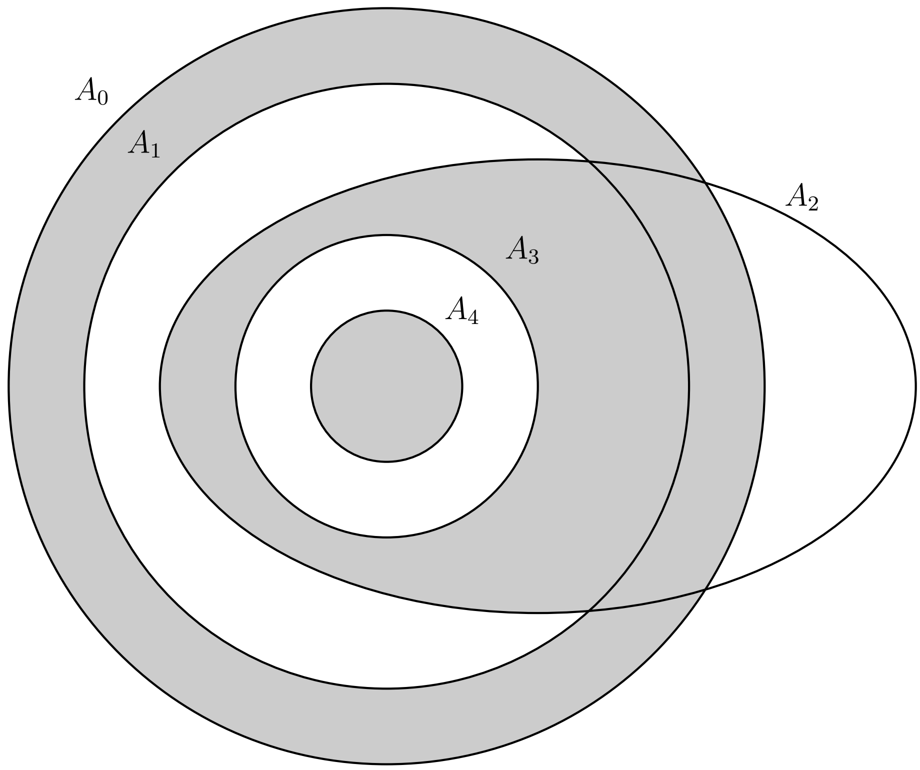

Figure 3.1: A $(\Pi^0_\alpha)_5$ set, where A_2 plays the same role as $A_2 \cap A_1$.

It is easy to see that $\bigcup_{n<\omega} (\Pi^0_\alpha)_n$ is exactly the set of the boolean combinations of Π^0_α sets. Moreover, this is a result of Kuratowski ([29]) that

$$\Delta^0_{\alpha+1} = \bigcup_{\beta<\omega_1} (\Pi^0_\alpha)_\beta,$$

the finite hierarchy being extendable to a transfinite one. This works because, as we saw in the proof of proposition 3.1.4, β differences of Π^0_α can give us infinite intersections of Π^0_α sets, but this, in particular, doesn't increase the complexity of our set since Π classes are closed under infinite intersections. On the other hand, if we had defined in the same way a hierarchy of differences for Σ sets, we would have obtained $(\Sigma^0_\alpha)_\omega = \Pi^0_{\alpha+1}$ because we would have allowed from them on, infinite unions of Σ^0_α sets. So we need to change the definition for the hierarchy of differences of Σ sets.

Definition 3.1.5 (Hierarchy of differences (for Σ^0_α sets))**.** *In any topology space X, given a natural number n and an ordinal number α, we say that a set A is $(\Sigma^0_\alpha)_m$ if there are Π^0_α sets $A_0, A_1, \dots A_{m-1}, A_m = X$ such that*

$$x \in A \leftrightarrow \textit{the smallest } i \textit{ such that } x \in A_i \textit{ is odd.}$$

We say that the sequence $\{A_i : i \leq m\}$ represents A (as an $(\Pi^0_\alpha)_m$ set).

Here we can assume that the sequence is nested upwards. However, both hierarchies are very close and for a given m are either complementary or equal, depending on the parity of m and thus if the last set is contained in the difference or not.

For determinacy, we can prove that the existence of winning strategies for $(\Pi^0_\alpha)_\beta$ payoff sets is equivalent to the existence of winning strategies with $(\Sigma^0_\alpha)_\beta$ payoff sets, for any ordinal numbers α and β. In the same way, it doesn't matter if we change "odd", to "even" in the definition, with regards to determinacy. One of the numerous proofs that can be done to show this claim is presented in [38] and the others are in the same vein.

An example we want to highlight is that at our level of complexity, determinacy in the Baire space, ω^ω, is equivalent to determinacy in the Cantor space, 2^ω. This is illustrated by the following proposition.

Proposition 3.1.6. *For any complexity of a class of definable sets* Γ *containing the* Δ^0_3 *sets, we have* $\Gamma\text{-}\mathsf{Det}^* \leftrightarrow \Gamma\text{-}\mathsf{Det}$. *In other words, determinacy for the class* Γ *of payoff sets is independent of whether we are in Baire space or Cantor space.*

Proof. Given $f\colon \omega \to \omega$ we encode it in 2^ω by

$$c_f = 0^{f(0)}10^{f(1)}10^{f(2)}\dots 10^{f(n)}\dots \qquad \text{so that}$$
$$f(n) = \hat{c_f} = m \leftrightarrow \exists k \left[c_f(k+1) = 1 \wedge c_f(k-m) = 1 \wedge \sum_{i=0}^{k+1} c_f(i) = n+1\right].$$

A strategy for G_A can thus be gotten effectively from one for $G_{\tilde{A}}$ with $\tilde{A} \subseteq 2^\omega$ such that

$$\begin{aligned} x \in \tilde{A} \leftrightarrow & [\exists n\ \forall (m > n)\ x(2m+1) = 0] \vee \\ & [(\forall n\ \exists (m > n)\ x(2m) = 1)\ \wedge\ (\hat{x} \in A)], \end{aligned}$$

which is a $\Delta^0_3 \cup \Gamma$ set. We added conditions on odd and even moves such that neither player have an interest to play infinitely many zeroes, which behaviour, if adopted by both players, would create a binary sequence that doesn't code any sequence of natural numbers.

Conversely, given any $A \subseteq 2^\omega$, a strategy for G_A can be found effectively from for $G_{\check{A}}$ with $\check{A} \subseteq \omega^\omega$ such that

$$x \in \check{A} \leftrightarrow (\exists n\ x(2n+1) \notin \{0,1\}) \vee x \in A,$$

which is a $\Sigma^0_1 \cup \Gamma$ set. Again we added a condition on odd moves such that player two cannot win too easily, by deciding to play a non-binary move and this way obviously create an element out of the payoff set. ☐

Although the exact strength of various determinacy axiom schemes is to this day fully characterised, as we already mentioned for Σ^0_1 determinacy ([49]) or even for, Δ^0_2, Σ^0_2, Δ^0_3, Σ^0_3 determinacy in terms of inductive definitions as showed in [18,37,51,52] by Hachtman, MedSalem and Tanaka, we still don't know what is the exact characterisation, in terms of reverse mathematics of $(\Pi^0_3)_m$ determinacy. What we know is that these axioms are stronger and stronger, reaching the edge of the provability from Z_2. This is what the theorem we want to expose now is about, from the same article of Montalbán and Shore [38].

Theorem 3.1.7 (Π^1_{m+2}-CA_0). *All $(\Pi^0_3)_m$ games are determined.*

First notice that, writing $S_{I,II}$ for $S_{I,II}(2^{<\omega}, X)$, $(\Pi^0_3)_m$-Det is the Π^1_3 sentence

$$\text{“}\forall X \in (\Pi^0_3)_m\text{”}\ [(\exists Y \in S_I)\ (\forall Z \in S_{II})\ Y \oplus Z \in X \quad \vee \\ (\exists Z \in S_{II})\ (\forall Y \in S_I)\ Y \oplus Z \notin X],$$

where $Z \oplus Y$ is identified with the play produced by player I and player II playing their respective strategies. Actually, the latter is an axiom scheme, where we should replace our quantification over $(\Pi^0_3)_m$ by an arbitrary $(\Pi^0_3)_m$ *formula* as it is defined for example in [45], according to definition 3.1.3. Hence, since strong Σ^1_{m+2}-DC_0 is Π^1_4 conservative over Π^1_{m+2}-CA_0 in virtue of 2.3.7 we will assume strong Σ^1_{m+2}-DC_0 as well as Π^1_{m+2}-CA_0.

In a similar idea to the one from 3.1.1, we want to unfold a fixed set A given by the finite nested sequence $\emptyset = A_m \subseteq \cdots \subseteq A_1 \subseteq A_0$ of Π^3_0 sets. To win the game, player I want to reach the exterior layers of the even sets while player II has to have the opposite behaviour. We set up the construction of the A_i for $0 \leq \mathit{player}\mathrm{I} < m$ as

$$A_i = \bigcap_{k<\omega} A_{i,k} \qquad \text{and} \qquad A_{i,k} = \bigcup_{j<\omega} A_{i,k,j},$$

with Σ^0_2 sets $A_{i,k}$ and closed sets $A_{i,k,j}$. In the following, we will consider sequences $s \in \omega^{\leq m}$ and trees $T \subseteq 2^\omega$. Given s we put $l \coloneqq m - |s|$.

Definition 3.1.8. *We define $\Sigma^1_{|s|+2}$ relations $P^s(T)$ by induction on $|s| \leq m$:*

- *When $|s| = 0$, $P^{\langle\ \rangle}(T)$ iff*

$$\textit{player I (resp. II) has a winning strategy in } G(T, A) \textit{ if } l \textit{ is even (resp. odd).} \tag{3.1}$$

- *For $|s| = n+1$ and l even, $P^s(T)$ iff there is a quasistrategy U for player I in S such that*

$$[U] \subseteq A \cup A_{l,s(n)} \qquad \textit{and} \qquad P^{s[n]}(U) \textit{ fails.} \tag{3.2}$$

- *For $|s| = n+1$ and l odd, $P^s(T)$ iff there is a quasistrategy U for player* II *in S such that*

$$[U] \subseteq \bar{A} \cup A_{l,s(n)} \qquad \text{and} \qquad P^{s[n]}(U) \text{ fails.} \tag{3.3}$$

A quasistrategy U witnesses $P^s(T)$ if U is as required in the appropriate clause, the latter being a $\Pi^1_{|s|+1}$ sentence.

Definition 3.1.9 (local witness)**.** *A quasistrategy U* locally witnesses *$P^s(T)$ if $|s| = n+1$ and U is a quasistrategy for player* I *(resp.* II*) if l is even (resp. odd) and there is $D \subseteq T$ such that, for every $d \in D$, there is a quasistrategy R^d for player* II *(resp.* I*) if l is even (resp. odd) in T_d such that the following conditions are satisfied:*

1. *$\forall d \in D \cap U,\ U_d \cap R^d$ witnesses $P^s(R^d)$.*
2. *$[U] \setminus \bigcup_{d \in D}[R^d] \subseteq A$ (resp. $\bar{A}$).*
3. *$\forall p \in S\ \exists^{\leq 1} d \in D,\ d \subseteq p \wedge p \in R^d$.*

We observe that "U locally witnesses $P^s(T)$" is a $\Sigma^1_{|s|+2}$ sentence.

The following lemma will be useful in a recursion in lemma 3.1.11 and will make us more familiar with the clauses of the preceding definition. It tells us that if a local witness is not a witness for the second reason, then we can construct a local witness for a preceding property.

Lemma 3.1.10. *Let $|s| = n+1 > 1$, if U locally witnesses $P^s(T)$ and $P^{s[n]}(T)$ is witnessed by some $\hat{T}$, then there is a local witness $\hat{U}$ of $P^{s[n-1]}(\hat{T})$ if $n > 1$. When $n = 1$, $P^{s[n]}(U)$ fails.*

Proof. WLOG, let $m - n (= m - |s[n]|)$ being odd. So $\hat{T}$ is a player II's quasistrategy (we can suppose $\hat{T} \subseteq U$). Suppose $n > 1$.

The main goal for $\hat{U}$ is to escape from the range of each R^d. Let $d \in \hat{D}$ iff $d \in \hat{T} \cap D$ and player II has a winning strategy in $G(\hat{T}_d, [\bar{R^d}])$, a Σ^1_2 set. For $d \in \hat{D}$ we let $\hat{R}^d$ be player II's non-losing quasistrategy in this game and $R^d = \emptyset$ for $d \in D \setminus \hat{D}$. The quasistrategy of player II is a Π^1_2 set as discussed in remark 1.2.11. Since $d \in D$ is a Σ^1_2 condition, the collection $\{(r, d) : d \in \hat{D} \wedge r \in \hat{R}^d\}$ exists by Π^1_3-CA_0. The idea is now that either player I can get out of a given R^d, or he has to get out of $\hat{R}^d$, so that, by definition, player I gets a strategy to go out of R^d.

By hypothesis $[\hat{R}^d] \subseteq [\hat{T}] \subseteq \bar{A} \cup A_{m-n,s(n-1)}$, so $\hat{R}^d$ satisfies the first condition to witness $P^{s[n]}(U_d \cap R^d)$. However, by property (1) of the local witness, $U_d \cap R^d$ witnesses $P^s(R^d)$ and so, in particular, $P^{s[n]}(U_d \cap R^d)$ fails and then $\hat{R}^d$ is not a witness for it. As a consequence, the second condition must fail with $\hat{R}^d$, that is there is a witness $\hat{U}^d$ for $P^{s[n-1]}(\hat{R}^d)$. We then define an indexed sequence $(\hat{U}^n)_{n\in\mathbb{N}}$ such that

$$\forall n\ \eta(n, \hat{U}^n), \quad \text{where } \eta(n, \hat{U}^n) := n \in D \to \hat{U}^n \text{ witnesses } P^{s[n-1]}(\hat{R}^d),$$

by $\Sigma^1_{|s|}$-AC_0. Finally, we similarly choose strategies for player I $\sigma_{p,d}$, winning in $G(\hat{T}_p, [\bar{R^d}])$ for $d \in \hat{D}$ when some $p \notin \hat{R}^d$ is reached.

We now (arithmetically in the above parameters) define by the following a quasistrategy $\hat{U}$ for player I in $\hat{T}$.

(i) If $p \in \hat{U}$ and there is no $d \in D$ such that $d \subseteq p$ and $p \in R^d$, then the child of p in $\bar{U}$ are the same as those in $\hat{T}$, otherwise;

(ii) If $p \in \hat{U}$ is a minimal extension of some $d \in D$ such that $p \in R^d \setminus \hat{R}^d$, then we escaped player II's non-losing strategy, which means that player I can play $\sigma_{p,d}$ until she reaches a $p \notin R^d$, otherwise;

(iii) If $p \in \hat{U} \cap \hat{D}$, let $\hat{U}_p = \hat{U}^p$ as long as we stay in $\hat{R}_d$.

We now prove the three conditions of local witnessing.

1. Take $p \in \hat{U} \cap \hat{D}$, by (iii) $\hat{U}_p \cap \hat{R}_d = \hat{U}^p$, which is witnessing $P^{s[n-1]}(\hat{R}^d)$.
2. Any play $x \in [\hat{U}] \setminus [\hat{R}^d]$ would have escaped R^d in some finite position by (ii). Thus

$$[\hat{U}] \setminus \bigcup_{d\in D} [\hat{R}^d] \subseteq [U] \setminus \bigcup_{d\in D} [R^d] \subseteq A,$$

   by hypothesis.
3. This condition is immediate from the corresponding hypothesis since we have just restrained D and the R^d's.

Finally when $n = 1$, we suppose for a contradiction that there exists a $\hat{T}$ such as for the case $n > 1$. We can then keep the same construction with the following differences. When we choose a witness $\hat{U}^d$ for $P^{s[n-1](\hat{R}^d)}$, we must take a winning strategy for player I in $G(\hat{R}^d, A)$ and we need to show that $[\hat{U}] \subseteq A$ to see that it witnesses $P^{\langle\ \rangle}(\hat{T})$ for the desired contradiction. The point here is that if we stay in some R^d, then we follow U^d, which is a winning strategy for player I in $G(\hat{R}^d, A)$. If we leave $\hat{R}^d$, then we leave R^d by (ii) in the definition of $\hat{U}$. If we leave every R^d, then we follow $\hat{T}$ and then stay in U and also wind up A by clause (2) of the definition of U being a local witness for $P^s(T)$. □

Now that the above construction has been done we can prove that there is no "local-only" witness.

Lemma 3.1.11. *If U locally witnesses $P^s(T)$, then U witnesses $P^s(T)$.*

Proof. WLOG, we suppose that l is even. Let us show the first property of the witness. Consider $x \in [U]$. If $x \in A$ there is nothing to prove. If not, by property (2) of the local witness, $x \in [R^d]$ for some $d \in D$. Then, by (1), $U_d \cap R^d$ witnesses $P^s(R^d)$ and so by the first property of the latter witness, $x \in A_{l,s(n)}$ as required.

We now show the second part of the definition by induction on $|s| = n+1 \leq m$. WLOG, we suppose m odd. We begin with $n = 0$. Suppose for a contradiction $P^{\langle\,\rangle}(U)$, that is there is a winning strategy τ for player II in $G(U, A)$.

We claim that there is a $d \in D$ belonging to τ such that every $x \supseteq d$ in $[\tau]$ is also in $[R^d]$. Suppose the contrary: $\forall d \in D\ \exists d \subset x \in [\tau] \setminus [R^d]$. Now note that every position

$$e \in \tau \setminus \bigcup_{d \in D, d \subset e} R^d,$$

has a minimal extension $\hat{d} \in D \cap \tau$. Otherwise, for any $e \subset x \in [\tau]$ we would have $x \notin \bigcup_{d \in D}[R^d]$. By property (2) of the local witness it would then follow that $x \in A$, a contradiction with our choice of τ. Next note that, by our assumption, any such $\hat{d}$ has a minimal extension $\hat{e} \in \tau \setminus R^{\hat{d}}$. By property (3) of the local witness, no $\hat{d} \subset e' \subset \hat{e}$ is in D and so $\hat{e}$ has the same property than e. We can iterate this process to create a sequence $e_j \subseteq \tau$ such that $\bigcup e_j = x \notin \bigcup_{d \subset x, d \in D} R^d$, which leads, as above, to a contradiction.

So we have such d. Thus, τ_d is a winning strategy for player II in $G(U_d \cap R^d, A)$, that is, $P^{\langle\,\rangle}(U_d \cap R^d)$ contradicting property (1) of the local witness and so establishing the desired property.

Now suppose $s = n + 1 > 1$. If $n = 1$, lemma 3.1.10 gives the conclusion. If $n > 1$, suppose for a contradiction that $P^{s[n]}(T)$ is witnessed by some $\hat{T}$. By applying lemma 3.1.10 we get a local witness $\hat{U}$ of $P^{s[n-1]}(\hat{T})$. Then the induction hypothesis implies that $\hat{U}$ is a witness for the same property, contradicting the existence of $\hat{T}$ and concluding our induction. □

Definition 3.1.12 (Failure everywhere)**.** *We say that $P^s(T)$* fails everywhere *if $P^s(T_p)$ fails for every $p \in S$. This is a $\Pi^1_{|s|+2}$ sentence.*

Lemma 3.1.13. *If $P^s(T)$ fails, then there is a quasistrategy W in S such that $P^s(W)$ fails everywhere.*

Proof. WLOG, we suppose l to be odd. First, if $|s| = 0$, then player II does not have a winning strategy. Then as we are used to, we define W to be player I's non-losing quasistrategy and verify that $P^s(W)$ fails everywhere.

Now suppose $|s| = n+1$ and, WLOG that l is even. Invoking $\Pi^1_{|s|+2}$-CA_0 let D be the set of the annoying d's, i.e.

$$d \in D \leftrightarrow d \in S \wedge P^s(T_d) \wedge \neg P^s(T_{d[|d|-1]}),$$

that is the minimal such d, an intersection of a $\Pi^1_{|s|+2}$ and a $\Sigma^1_{|s|+2}$ set. We suppose D to be non-empty and, as we often do now, use $\Sigma^1_{|s|+2}$-AC_0 to chose a sequence of witnesses U^d of $P^s(T_d)$ for each $d \in D$.

Consider now the game $G(T, B)$ where $B = \{x \in [T] \mid \exists d \in D \; d \subseteq x\}$. We claim that player I has no winning strategy in this game. If there were one σ, then we could define a quasistrategy U for player I in T by following σ until a position $d \in D$ is reached, at which point we move into U^d. With D and $R^d = T_d$ we can easily verify that three clauses of U locally witnessing $P^s(T)$ are satisfied:

1. Take $d \in D \cap U$, $U_d \cap R^d = U^d$ witness $P^s(T_d)$;
2. Since $[U] \subseteq \bigcup_{d\in D}[U^d]$, $[U] \setminus \bigcup_{d \in D}[T_d] = \emptyset$;
3. Taking any $p \in T$ if there exists $d \subseteq p$ such that $p \in R^d$ the unicity follows from the minimality of d;

which lead, by lemma 3.1.11, to a contradiction with the fact that $P^s(T)$ fails.

Thus, we let W be player II's non-losing quasistrategy in $G(T, B)$ and σ_p be a chosen winning strategy for player I if $p \in S \setminus W$ is reached. Suppose for a contradiction that W is not as required. Then for some $q \in W$ we can find a witness $\hat{U}$ of $P^s(W_q)$. Consequently, we define a quasistrategy U for player I in T_q:

(i) We begin to set up $U \cap W_q = \hat{U}$;

(ii) If $p \in U \setminus W$, player I plays σ_p until she reaches a position $d \in D$ from where she plays U^d, witnessing $P^s(T_d)$.

If we now consider U, $\hat{D} := D \cup \{q\}$, $R^d := T_d$ and $R^q = W_q$, we verify that U locally witnesses $P^s(T_q)$:

1. Take $d \in \hat{D} \cap U$, $U_d \cap T_d$ witnesses $P^s(T_d)$ and $U \cap W_q = \hat{U}$, $P^s(W_q)$;
2. As before, $[U] \setminus \bigcup_{d \in D}[R^d] \subset \emptyset$;
3. Again it follows from minimality and the fact that $q \in W$, which is non-losing.

Using lemma 3.1.11 we know that $P^s(T_q)$ holds, but then by definition of D there is a minimal $d \subseteq q$ in D, which contradicts the choice of W. Thus $P^s(W)$ fails everywhere. □

Definition 3.1.14 (Strong witness)**.** *For $|s| = n+1$, W strongly witnesses $P^s(T)$ if, for all $p \in W$, W_p witnesses $P^s(T_p)$, that is, W witnesses $P^s(T)$ and $P^{s[n]}(W)$ fails everywhere. This is a $\Pi_{|s+1|^1}$ sentence.*

The following lemma is straightforward from the definition and the last lemma about properties failing everywhere.

Lemma 3.1.15. *If $P^s(T)$, then there is a W that strongly witnesses it.*

Now we prove the principal result about the property P, and this is the last one we need to prove theorem 3.1.7.

Lemma 3.1.16. *If $|s| = n+1$, then at least one of $P^s(T)$ and $P^{s[n]}(T)$ holds.*

Proof. We prove the lemma by reverse induction on $n < m$. Suppose WLOG that $m - n$ is odd and $P^s(T)$ fails. Using Σ^1_{m+2}-DC_0, we define by induction on the length of positions a quasistrategy U for player II in S along with $D \subseteq S$ and R^d for $d \in D$ showing that

$$U \text{ locally witnesses } P^{s[n]}(T) \text{ if } n > 0 \quad \text{and} \quad U \text{ witnesses } P^{s[n]}(T) \text{ if } n = 0.$$

It suffices then to use lemma 3.1.11 to have the desired property in every case.

Initiation: $\langle\,\rangle \in U$, we say that it marks 0.

(i) If $n = m-1$, by lemma 3.1.13 we set $W^{\langle\,\rangle}$ be a quasistrategy for player II in S such that $P^s(W^{\langle\,\rangle})$ fails everywhere.

(ii) If $n < m-1$, then we know by reverse induction that $P^{s^\frown 0}(T)$ holds. Applying lemma 3.1.15 there exist a $W^{\langle\,\rangle}$ strongly witnessing this fact and so $P^s(W^{\langle\,\rangle})$ fails everywhere.

Recursion step: Take $q \in U$ marking $j < \omega$, with $P^s(W^q)$ failing everywhere. Consider the closed game

$$G(W^q, A_{m-n-1,s(n),j}).$$

If it is not a win for II, we put $q \in D$ and define $\hat{R}^q$ to be player I's non-losing quasistrategy in this game. We also define R^q to be $\hat{R}^q$ on W^q and to simply T_q elsewhere. Thus, $[\hat{R}^q] \subseteq A_{m-n-1,s(n),j} \subseteq A^{m-n-1,s(n)}$ by definition and since $\hat{R}^q$ is a non-losing quasistrategy for a closed set. Thus, if $P^{s[n]}(\hat{R}^q)$, the two properties of $\hat{R}^q$ witnessing $P^s(W^q)$ would be satisfied, contrary to our assumption that $P^s(W^q)$ fails everywhere. So we may take U^q to be a witness for $P^{s[n]}(\hat{R}^q)$ (a Π^1_s relation). We now continue to define U:

1. On $\hat{R}^q$, $U = U^q$;
2. If $p \notin \hat{R}^q$ ($p = q$ if the game is not a win for I), player II can follow a winning strategy τ_p until he reaches a q' with $[W^q_{q'}] \cap A_{m-n-1,s(n),j} = \emptyset$,

which one exists since player II is playing an open game. As a consequence, we say that q' marks $j+1$. Now $P^s(W^q_{q'})$ fails everywhere since $P^s(W^q)$ does.

(i) If $n = m - 1$, we define $W^{q'} = W^q_{q'}$.

(ii) If $n < m - 1$, then by our reverse induction on n, $P^{s^\frown j+1}(W^q_{q'})$ and there exists $W^{q'}$ strongly witnessing this fact, as well as $P^{s^\frown j+1}(T_{q'})$.

In the cases (ii), with $n < m - 1$, we have to choose strategies, however, each choice depends on the preceding ones. For this reason, we have to use the scheme of strong dependent choice, to have the existence of a sequence $Z = (Z_n)_{n\in\mathbb{N}}$, such that

$$\forall n\ \forall Y\ (\eta(n, Z^n, Y) \to \eta(q, Z^n, Z_n)),$$

where $\eta(n, Z^n, Y) \coloneqq Y$ strongly witnesses

$$\begin{cases} P^{s^\frown n}((Z_{n-1})_q) \text{ if } n > 0, \\ \qquad P^{s^\frown 0}(T) \text{ if } n = 0, \end{cases}$$

a $\Pi^1_{|s|+2}$ and so at worst Π^1_{m+1} and so Σ^1_{m+2} relation, where $Z_n = W^q$ for the q marking n as in our construction. Actually, Z should also contain the information about the U^q's but this does not increase the logical complexity of η.

If $n > 0$ we show the properties for U, together with D and R^d locally witnessing $P^{s[n]}$.

1. Take $d \in D \cap U$, by construction $U_d \cap R^d = U_d \cap \hat{R}^d = U^d$ and U^d witnesses $P^{s[n]}(\hat{R}^d)$;
2. We prove it here under;
3. It follows from the fact we put a new $d \in D$ only once we have left $\hat{R}^d$.

Let $x \in [U]$ and

$$\emptyset = q_0 \subset q_1 \subset \cdots \subset q_i \subset \ldots$$

be the strictly increasing sequence of the initial segments q of x such that q_j marks j. By construction, each $q_j \in D$. If the sequence terminates at some $q = q_k$, then, by definition, x never leaves $\hat{R}^d$ and so $x \in R^d$. So if x is out of the R^d's, the sequence is infinite and

$$x \notin A_{m-n-1} \subset A_{m-n-1,s(n)} = \bigcup_{j<\omega} A_{m-n-1,s(n),j}.$$

If $n + 1 = m$, $x \notin A_0$ implies $x \notin A$ and we are done. If $n + 1 < m$, as W^{q_j} witnesses $P^{s^\frown j}(T_{q_{j+1}})$ and $m - |s^\frown j|$ is odd,

$$x \in \bar{A} \cup \bigcup_{j<\omega} A_{m-n-2,j} = \bar{A} \cup A_{m-n-2}.$$

As, by our case assumptions, $m-n-1$ is even, it follows that $x \in A_{m-n-2} \setminus A_{m-n-1} \subseteq \bar{A}$. By lemma 3.1.11, U witnesses $P^{s[n]}$.

Finally, if $n = 0$, then we argue that U is a winning quasistrategy for player II in $G(T, A)$. Consider any $x \in [U]$. If there is a $d \in D$ such that $x \in [\hat{R}^d]$, then $x \in U^d$ by construction. Now U^d is a witness for $P^{\langle\ \rangle}(\hat{R}^d)$ (as $n = 0$, $s[n] = \langle\ \rangle$), that is, U^d is a winning strategy for player II in $G(\hat{R}^d, A)$. Thus $x \in \bar{A}$, as required. On the other hand, if x leaves every $\hat{R}^d$, then, by the argument above, $x \in \bar{A}$ as well. □

Proof of theorem 3.1.7. WLOG, we suppose that m is odd and player II has no winning strategy in $G(T, A)$, that is $P^{\langle\ \rangle}(T)$ fails. By lemma 3.1.13, there is a quasistrategy $W^{\langle\ \rangle}$ that player I can follow such that $P^{\langle\ \rangle}(W^{\langle\ \rangle})$ fails everywhere. We define a quasistrategy U for player I in $W^{\langle\ \rangle}$ by induction on $|p|$ for $p \in U$.

To $\langle\ \rangle \in U$, we associate the quasistrategy $W^{\langle\ \rangle}$ which fail to $P^{\langle\ \rangle}(W^{\langle\ \rangle})$ everywhere. Suppose then $p \in U$, $|p| = j+1$ and W^p has been defined with $P^{\langle\ \rangle}(W^p)$ failing everywhere. The child q of p in U are the same as those of p in W^p. Since $P^{\langle\ \rangle}(W^p)$ fails everywhere, so does it for W^p_q and by lemma 3.1.16, it implies $P^{\langle j \rangle}(W^p_q)$. Now we use of lemma 3.1.15 to get the existence of a W^q that strongly witnesses it. To continue our induction, we have to choose such a W^q, which depends on the previously chosen ones, we can do it since we dispose of strong Σ^1_3-DC_0, the same way as exposed in the proof of the preceding lemma.

Now we show that U is winning, giving rise to a winning strategy for player I who can play the minimal move each time she has to choose. Consider any play $x \in [U]$. By construction, for every j

$$x \in [W^{x[j+1]}] \qquad \text{and} \qquad W^{x[j+1]} \text{ witnesses } P^{\langle j \rangle}(W^{x[j]}_{x[j+1]}).$$

By the first property of the witness, for every j

$$x \in A \cup A_{m-1,j} \quad (\text{resp. } \bar{A} \cup A_{m-1,j}).$$

As $\bigcap_{j<\omega} A_{m-1,j} = A_{m-1} \subseteq A$ (resp. $\bar{A}$) by definition, it follows that U is winning for player I (resp. II) in $G(A, T)$, as desired. □

This result witness that determinacy enlarges the field of reverse mathematics, since usually, all theorems provable in Z_2 are equivalent to one of the big five and few are known to be stronger than Π^1_1-CA_0. (one of the few example, for Π^1_2-CA_0 is studied in [42]). As we are about to see, we can't hope for any reversal of this theorem as from the same paper [38], it is proved that Δ^1_{m+2}-CA_0 does not prove $(\Pi^0_3)_m$-Det and even "$\forall n$ $(\Pi^0_3)_m$-Det" does not prove Δ^1_2-CA_0. One can find further analysis about that kind of limitative results, notably by the works of Montalban and Shore, Pachecho and Yokoyama and Welch in [39, 45, 53].

3.2 Failure of Determinacy in Subsystems of Z_3

As well as when Martin proved in [34] that Borel determinacy does require the axiom of replacement, showing that ZC was not sufficient to prove Borel determinacy, we could wonder how much of determinacy remains when we only allow moderate use of the power set axiom. Such a result was first proved by Friedman [15], who showed that they are Σ^0_5 countable games such that ZFC^- is not powerful enough to prove their determinacy that is, the existence of winning strategies for all of them. We will first present a sharper result of Martin. To this aim, we need a way to construct a canonical model from a given complete, consistent theory, extending our setup.

Definition 3.2.1 (The term model)**.** *Let T be a complete theory in the language of set theory extending some sub-theory of ZFC^- and satisfying* Constructibility*. The* term model

$$A = (|A|, \in_A)$$

of T is defined as follows. Two formulae $\phi(v)$ and $\psi(v)$ will be T-equivalent if within T

- *The set v^* being the $<_L$-least v such that* $\begin{cases} v = 0 & \text{if } \forall v' \ \neg\phi(v'); \\ \phi(v) & \text{otherwise,} \end{cases}$
- *The set w^* being the $<_L$-least w such that* $\begin{cases} w = 0 & \text{if } \forall w' \ \neg\phi(w'); \\ \psi(w) & \text{otherwise,} \end{cases}$
- $v^* = w^*$.

We then put $|A|$ as the sets of such equivalence classes, labelled by their respective representative and we define $\in_A$ as the $\in$ relation between representatives.

This is a standard way to construct models out of complete theories, similar to the proof of Gödel's completeness theorem as done in [12, 48]. The following result is an exercise in [35] and a theorem in [17] which will serve as a warm-up for the result we want to show.

Theorem 3.2.2. *The determinacy of all Σ^0_4 countable games is not provable in ZFC^-.*

Proof. Let β_0 be the minimal ordinal such that $\mathrm{L}_{\beta_0} \models \mathsf{ZFC}^-$.

Claim 1: There is no $a \subseteq \omega$ such that $a \in L_{\beta_0+1} \setminus L_{\beta_0}$ and β_0 is the least ordinal with this property.

This is a consequence of theorem 2.1.8 from which it follows that β_0 is also the first ordinal such that $L_{\beta_0} \models \mathsf{Z}^- + \Sigma_n$ REPLACEMENT, for every $0 \leq n$. Then, for every ordinal $\alpha < \beta_0$ such that $L_\alpha \models$ INFINITY, some amount of separation have to fail.

We define a Σ^0_4 game G in $2^{<\omega}$ such that G is a win for player I but the set of Gödel numbers of sentences true in L_{β_0} is recursive uniformly in any strategy for player I in G. For each play x of this game, we set

$$T_{\mathrm{I}}(x) = \{\phi : x(2\sharp(\phi)) = 1\}$$
$$T_{\mathrm{II}}(x) = \{\phi : x(2\sharp(\phi) + 1) = 1\}.$$

Each player loses automatically if for any of them their set $T_i(x)$, $i = \mathrm{I}, \mathrm{II}$ does not correspond to the sentences true in an ω-model of ZFC^-+"$\mathrm{V} = \mathrm{L}_{\beta_0}$".

Claim 2: The latter is a Π^0_2 condition.

By "$\mathrm{V} = \mathrm{L}_{\beta_0}$", we mean, $V = L$ and $\forall\beta\ L_\beta \not\models \mathsf{ZFC}^-$. Obviously, saying that the axioms of ZFC^-+"$\mathrm{V} = \mathrm{L}_{\beta_0}$ are included in T_{I} or T_{II} is harmless while asking of them to be complete is Π^0_1. Requiring the term model $\mathcal{M}$ to be an ω one is the Π^0_2 condition

$$\forall i \in \omega_{\mathcal{M}}\ \exists n\ \mathcal{M} \models i = \sum_{j=0}^{n} 1.$$

Supposing this does not happen, the term models of $T_{\mathrm{I}}(x)$ and $T_{\mathrm{II}}(x)$ are then isomorphic to ω-models. Let $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ be such ω-models. Player I then wins if and only if one of the following holds:

1. The model $\mathcal{M}_{\mathrm{I}}$ is isomorphic to an initial segment of $\mathcal{M}_{\mathrm{II}}$;
2. There is an ordinal α of $\mathcal{M}_{\mathrm{I}}$ such that $\mathrm{L}^{\mathcal{M}_{\mathrm{I}}}_\alpha$ is isomorphic to an initial segment of $\mathcal{M}_{\mathrm{II}}$ but $\mathrm{L}^{\mathcal{M}_{\mathrm{I}}}_{\alpha+1}$ is not.

Claim 3: There is a fixed Σ^1_2 formula $\phi(X, Y)$ of second order arithmetic such that, given an ω-model $\mathcal{M}$ as required earlier, for $\alpha \in Ord(\mathcal{M})$ and $(\omega \supseteq) b \in L^{\mathcal{M}}_{\alpha+1} \setminus L^{\mathcal{M}}_\alpha$,

- $L^{\mathcal{M}} \cap \mathcal{P}(\omega) \models \exists Y\ \phi(b, Y)$;
- For all $c \in L^{\mathcal{M}}_{\alpha+1} \cap \mathcal{P}(\omega)$,

$$L^{\mathcal{M}} \cap \mathcal{P}(\omega) \models \phi(b, c) \leftrightarrow c \text{ codes a model } (\omega, E) \simeq L^{\mathcal{M}}_\alpha.$$

Let us first write the set-theoretic version of ϕ such that it works as intended. We will use the Δ_0^{Set} predicates func, ord, lim, bij to summarize the condition of being respectively a function, an ordinal, a limit (ordinal) and a bijection and define by $\mathrm{dom}(f)$, the domain of a function f.

$$\begin{gathered}\phi^{\mathrm{Set}}(x,y) \leftrightarrow \exists f\ \exists \alpha\ \mathrm{func}(f) \wedge \mathrm{ord}(\alpha) \wedge [\mathrm{dom}(f) = \alpha+1] \wedge \\ [f(0)=\emptyset]\ \wedge \forall(\gamma<\alpha)\ [f(\gamma+1) = \mathrm{Def}(f(\gamma)) \wedge \lim(\gamma) \to f(\gamma) = \bigcup_{\beta<\gamma} f(\beta)] \wedge \\ x \notin f(\alpha)\ \wedge x \in \mathrm{Def}(f(\alpha)) \wedge \\ \exists(j\colon \omega \to f(\alpha))\ \mathrm{bij}(j) \wedge [\forall m,n \in \omega\ ((m,n)\in y \leftrightarrow j(m) \in j(n))].\end{gathered}$$

Roughly, the formula is saying that there is some L_α, defined as the image of a function f (line 1-2). Moreover, L_α is uniquely determined by being the last step of the constructible hierarchy before being able to define x (line 3). We finally add that the membership relation has to be countably coded through the relation $y \subseteq \omega \times \omega$ (line 4). This is exactly the way the constructible hierarchy is defined in [48, VII.4].

It is clear that $\phi(x,y)^{\mathrm{Set}}$ is the desired formula, hence we may take its second-order translation, which is a Σ_2^1 formula $\phi(X,Y)$, according to theorem 1.3.16. Thus because of claim 1, the second-order part of $\mathcal{M}$ determines the isomorphism type of $\mathcal{M}$.

Finally, equality between subsets of ω is a Π_1^0 condition. So we may define $\mathcal{A} \subseteq \omega$ such as to code the set of isomorphic part of ω from $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ as

$$\begin{aligned}(z,w) \in \mathcal{A} \leftrightarrow \exists x \in \mathbb{R}_{\mathrm{I}}\ \exists y \in \mathbb{R}_{\mathrm{II}}(\mathcal{M}_{\mathrm{I}} \models x \text{ codes } z\ \wedge \mathcal{M}_{\mathrm{II}} \models y \text{ codes } w\ \wedge \\ \forall(n\in\omega)\ (\mathcal{M}_{\mathrm{I}} \models n \in x \leftrightarrow \mathcal{M}_{\mathrm{I}} \models n \in y)),\end{aligned}$$

which is thus a Σ_2^1 set. This way, in virtue of claim 3, condition 1 is just

$$\forall z \in \mathbb{R}_{\mathcal{M}_{\mathrm{I}}}\ \exists w\ (z,w) \in \mathcal{A},$$

which is Π_3^0. Similarly, we will express condition 2 as

$$\begin{gathered}\exists \alpha\ (\mathcal{M}_{\mathrm{I}} \models \text{“}\alpha \text{ is an ordinal number with successor } \alpha+1\text{”}) \\ [\forall b \in \mathbb{R}_{\mathcal{M}_{\mathrm{I}}}\ ((\mathcal{M}_{\mathrm{I}} \models b \in L_\alpha)\ \exists c \in \mathbb{R}_{\mathcal{M}_{\mathrm{II}}}\ (b,c)\in\mathcal{A}) \wedge \\ \forall d \in \mathbb{R}_{\mathcal{M}_{\mathrm{I}}}\ \forall e \in \mathbb{R}_{\mathcal{M}_{\mathrm{II}}}\ ((\mathcal{M}_I \models \phi(\alpha,d)) \to (d,e)\notin\mathcal{A})],\end{gathered}$$

which is a Σ_4^0 sentence.

Now that we have defined Π_4^0 game G, we prove that neither player can have a winning strategy contained in L_{β_0}. Indeed, G is a win for I who can play the set of sentences that are true in L_{β_0}. So now suppose that there is a winning strategy for I in L_{β_0} and that II is playing such as copying every move of I then the winning strategy will exactly be the set of sentences that are true in L_{β_0} since otherwise, from the very moment where I decide to deviate that strategy, supposing that II continue to play the theory of L_{β_0}, the so-said winning strategy of I would lead to the victory of the second player. However, $Th(L_\beta) \in L_\beta$ is a contradiction to theorem 2.1.7 since it defines truth of L_β inside L_β. $\square$

This result is sharpened by Montalbán and Shore in [38]. Assuming constructibility, ZFC^- is equivalent to being n-admissible for all n, that is to the theory

$$\mathsf{KP} + \bigcup_{1 \leq n < \omega} (\Delta_{n-1} \text{ COLLECTION} + \Sigma_{n-1} \text{ SEPARATION})$$

does not prove the determinacy of Π^0_4 games, since it is false in some models of the theory, as we just showed. The result of Montalbán and Shore is a local version of this theorem, leading to a finer theorem of unprovability.

Theorem 3.2.3 ([38]). *For all $2 \leq n < \omega$,*

$$\mathsf{KP} + \Delta_{n-1} \textit{ COLLECTION} + \Sigma_{n-1} \textit{ SEPARATION} \nvdash (\Pi^0_3)_{n-1}\text{-}\mathsf{Det}$$

By the translation in second-order arithmetic from the theorem 2.2.11 and theorem 3.1.7 of the preceding section, Montalbán and Shore thus proved that

$$\Delta^1_{n+2}\text{-}\mathsf{CA}_0 < (\Pi^0_3)_n\text{-}\mathsf{Det} < \Pi^1_{n+2}\text{-}\mathsf{CA}_0,$$

for $1 \leq m$, in terms of provability power along the strength of the determinacy scale.

On the other hand, Martin presented results in [35, 2.3] showing

$$\mathsf{ZFC}^- + \text{“}\mathcal{P}^n(\omega) \text{ exists”} < \Pi^0_{n+4}\text{-}\mathsf{Det} < \mathsf{ZFC}^- + \text{“}\mathcal{P}^{n+1}(\omega) \text{ exists”},$$

for all $n < \omega$ (with $\mathcal{P}^0(\omega) = \omega$). Our goal is now to generalize the proof of Montalbán and Shore (theorem 3.2.3), according to the first inequality of Martin, in the following way.

Theorem 3.2.4. *For all $2 \leq n < \omega$,*

$$\mathsf{KP}^2_n := \mathsf{KP} + \text{“}\mathcal{P}(\omega) \textit{ exists”} + \Delta_{n-1} \textit{ COLLECTION} + \Sigma_{n-1} \textit{ SEPARATION} \nvdash (\Pi^0_4)_n\text{-}\mathsf{Det}.$$

We will see that indeed, when going beyond the countable case, we need an additional condition compared to the theorem of Montalbán and Shore. Along the way we introduced the notation KP^i_n where the subscript $1 \leq n$ denotes the amount of SEPARATION and COLLECTION and the superscript $0 \leq i$ reminds the additional condition “$\mathcal{P}^{(i-1)}(\omega)$ exists”, when $1 \leq i$. For example, KP is KP^0_1 ; KP+ INFINITY is KP^1_1 (and can be interpreted in the realm of second-order arithmetic) ; a model of KP^0_n is a n-admissible set, etc. With the theories, KP^2_n, we enter the realm of third-order arithmetic. For a more precise study and axiomatisation of third-order arithmetic, in the continuation of Z_2, one can consult [19].

For the proof, we will proceed to build up a game in $2^{<\omega}$ with a $(\Pi^0_4)_n$ winning condition. However, to keep the way we describe the game the most positive and intuitive possible, we will describe this winning condition by its complementary, that is, n Σ^0_4 conditions $\phi_{i=0,\ldots,n-1}$ which will lead to the victory of player I if and only if the smallest i such that ϕ_i doesn't hold is even (keep in mind we add the condition $\phi_n(m) \equiv m \neq m$ at the end). Otherwise player II wins. Since player I's goal is that the first condition to fail is even, we will stack in even conditions what we will call the $C_{\mathrm{II}}i$'s, that is what player II should play to avoid losing. Similarly, the $C_{\mathrm{I}}i$'s will appear in odd conditions, to be satisfied by I(at least before the failure of an even condition). We follow as a guide, the proof of theorem 3.2.3 from [38].

The model that will witness the failure of $(\Pi^0_4)_{m-1}$ determinacy is a (the unique) well-founded one of the theory

$$T^1_n \coloneqq \mathsf{KP}^2_n + V = L + \forall\alpha \in Ord(L_\alpha \text{ is not a model of } \mathsf{KP}^2_{\mathsf{n}}).$$

The later model is of course $L_{\alpha^1_n}$, where α^1_n is the smallest ordinal α such that L_α is a model of KP^2_n.

We begin to define the first conditions $C_{\mathrm{I}}0$ and $C_{\mathrm{II}}0$.

$$\begin{aligned}
(C_{\mathrm{I}}0): &\qquad \mathcal{M}_{\mathrm{I}} \models T^1_n \wedge \mathcal{M}_{\mathrm{I}} \text{ is an } \omega\text{-model } \wedge \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{I}}}) \nsubseteq \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{II}}}).\\
(C_{\mathrm{II}}0): &\qquad \mathcal{M}_{\mathrm{II}} \models T^1_n \wedge \mathcal{M}_{\mathrm{II}} \text{ is an } \omega\text{-model } \wedge \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{II}}}) \nsubseteq \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{I}}}).
\end{aligned}$$

By the construction of the difference hierarchy, since we are looking for the first condition to fail (and we will see at the end that all can't hold), we will suppose when stating later conditions that the preceding ones are all satisfied so far. In particular, from now on both term models can't be well-founded since the only well-founded ω-model of T^1_n is $L_{\alpha^1_n}$.

The conditions we want now to define have the purpose to detect which player played the theory of an ill-founded model to make it lose. Since the best chance for a player to win is then to play the theory of $L_{\alpha^1_n}$, we will suppose that one of the models is well-founded, and call it $\mathcal{M}$, and then that the other is ill-founded, and denote it by $\mathcal{N}$.

We define the larger common segment (identifying it to the actual isomorphism between them), that is, the well-founded part of $\mathcal{N}$, as follows. First we define its second order version, for $x_0 \in \mathbb{R}_{\mathcal{M}}$ and $y_0 \in \mathbb{R}_{\mathcal{N}}$,

$$(x_0, y_0) \in \mathcal{A}^0 \leftrightarrow \forall n \in \omega(\mathcal{M} \models n \in x_0 \leftrightarrow \mathcal{N} \models n \in y_0).$$

Now let us move on to the third-order common part, with a first attempt at a definition that we will have to sharpen later.

$$(z,w) \in \mathcal{A}^1 \leftrightarrow \exists x_1 \in \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{I}}}), y_1 \in \mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\mathrm{II}}})\ (\mathcal{M}_{\mathrm{I}} \models x_1 \text{ codes } z \wedge \mathcal{M}_{\mathrm{II}} \models y_1 \text{ codes } w \wedge \\ \forall x_0, (x_0,y_0) \in \mathcal{A}^0 \rightarrow (\mathcal{M}_{\mathrm{I}} \models x_0 \in x_1 \leftrightarrow \mathcal{M}_{\mathrm{II}} \models y_0 \in y_1)). \tag{3.4}$$

which is a Σ^0_3 set. Nevertheless, if $\mathcal{M}$ and $\mathcal{N}$ don't have the same reals it could be that some elements of $\mathcal{A}^1$ aren't actually the same set of real numbers. To avoid that, we could add conditions like

$$\forall x_0 \in \mathbb{R}_{\mathcal{M}_{\mathrm{I}}}(\mathcal{M}_{\mathrm{I}} \models x_0 \in x_1) \rightarrow \exists y_0\ (x_0,y_0) \in \mathcal{A}^0.$$

However, this would increase the complexity of $\mathcal{A}^1$ to the one of a Σ^0_4 set. Instead, we will add preliminary conditions $C_{\mathrm{I}}1$ and $C_{\mathrm{II}}1$ that will allow us to redefine accurately the third-order common part.

On the other hand, the point of defining the following conditions will be to force each player to exhibit more and more of their respective unique elements and check that these do not form a descending sequence, proving the model to be ill-founded. Along the way we want to show that these conditions imply L_α, the well-founded part of $\mathcal{N}$, to be a model of KP^2_i for $i \leq n$ so that all conditions cannot be satisfied (by definition of α^1_n). For that to work, we need to ensure that so far L_α, the common well-founded part, is a model of "$\mathcal{P}(\omega)$ exists". To do that we will need a result like the one in [17], characterising constructible ill-founded models of set theory. We cite it briefly because its proof requires notions we didn't introduce in the present study.

Lemma 3.2.5 (Overspill)**.** *Let $\mathcal{N}$ be an ω-model of $V = L$ and suppose $\mathcal{N}$ is ill-founded with* $\mathrm{wfo}(\mathcal{N}) = \alpha$ *and that $\kappa \in L_\alpha$ is the largest cardinal of L_α. Say $X \in \mathcal{N}$ is a non-standard code if $X \subseteq \kappa$ codes a linear order of κ so that $\mathcal{N}$ has an isomorphism from X onto some non-standard ordinal of $\mathcal{N}$. Then,*

$$\{X \in \mathcal{N} \setminus L_\alpha \mid X \textit{ is a non-standard code } \}$$

is non-empty, and has no $<^{\mathcal{N}}_L$-least element.

Thus, by requiring that this set, for $\kappa = \omega$, is either empty or has a $<^{\mathcal{N}}_L$-least element, we have that κ cannot be the greatest cardinal of L_β. From there, since we have a cardinal ω_1 in L_α, we can show that the collection of the set of natural numbers of L_α is a set of L_α and this way that it is a model of "$\mathcal{P}(\omega)$ exists". However, since we don't know which one of the player play the ill-founded model, we have to state the condition in both models to ensure this fact.

$(C_{\text{I}}1):\quad \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} \setminus \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \neq \emptyset \to \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} \setminus \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}}$ has a $<_L^{\mathcal{M}_{\text{I}}}$-least element.

$(C_{\text{II}}1):\quad \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \setminus \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} \neq \emptyset \to \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \setminus \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}}$ has a $<_{\text{II}}^{\mathcal{M}_{II}}$-least element.

In his proof of theorem 3.2.2, Hachtman proves that these are Σ_4^0 conditions. While in his case it was used to detect which of the model was ill-founded, since, provably in the model, everything was countable, it simply tells us that the descending sequence of $\mathcal{N}$ cannot be too simple. So as soon as both $(C_{\text{I}}0), (C_{\text{I}}1)$ and $(C_{\text{II}}0), (C_{\text{II}}1)$ are satisfied, L_α is a model of "$\mathcal{P}(\omega)$ exists".

What is more, we can now better handle the situation where $\mathcal{M}$ and $\mathcal{N}$ don't have the same reals. There are four cases:

1. If $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} = \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}}$ (a Π_3^0 condition), we define $\mathcal{A}^1$ as in 3.4;
2. If $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} \subsetneq \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}}$ (a Δ_4^0 condition), then we know by condition $(C_{\text{II}}1)$ that there exists a $<_L^{\mathcal{M}_{\text{II}}}$-least element of $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \setminus \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}}$ and thus a least δ such that $L_\delta^{\mathcal{M}_{\text{II}}}$ contains that element. Then we define $\mathcal{A}^1$ as in 3.4 but with $L_\delta^{\mathcal{M}_{\text{II}}}$ instead of $\mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\text{II}}})$;
3. The case $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \subsetneq \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}}$ is similar to the preceding one;
4. If $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}} \nsubseteq \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}}$ and $\mathcal{P}^1(\omega)_{\mathcal{M}_{\text{II}}} \nsubseteq \mathcal{P}^1(\omega)_{\mathcal{M}_{\text{I}}}$ (a Σ_3^0 condition), then we have minimal δ_1 and δ_2 such that all the reals defined below them in the constructible hierarchy are common to both model. Then we define $\mathcal{A}^1$ as in 3.4 but with $L_{\delta_1}^{\mathcal{M}_{\text{I}}}$ and $L_{\delta_2}^{\mathcal{M}_{\text{II}}}$ instead of $\mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\text{I}}})$ and $\mathcal{P}^1(\mathbb{R}_{\mathcal{M}_{\text{II}}})$ respectively.

In all cases, the definition of $\mathcal{A}_1$ will remain Σ_3^0. Since the remaining conditions will always make use of $\mathcal{A}^1$, we will write $C_{::}k$ (for $2 \leq k \leq n-1$) as one condition depending of $\mathcal{A}^1$, but it is actually a conjunction of four conditions "$H_i \to C_{::}k(\mathcal{A}_i^1)$", with $i \in \{1,2,3,4\}$ for the four cases, denoted by H_i. Then we see that if $C_{::}k$ is Σ_4^0, the whole conjunction is still Σ_4^0. For instance in H_2 we would have the Δ_4^0 condition plus

$$\exists\delta\ \forall(\gamma<\delta)\ \forall a \in \mathcal{P}^1(\omega)_{\mathcal{M}_{II}}\ (a \in L_\gamma^{\mathcal{M}_{\text{II}}} \to \exists b\ (a,b) \in \mathcal{A}^0),$$

which is Σ_4^0, but we will have to put "$\exists\delta$" in front of the implication "$H_2 \to C_{::}k(\mathcal{A}_2^1)$" since $\mathcal{A}_2^1$ depends of it, thus making of the whole condition a Σ_4^0 sentence if $C_{::}k(\mathcal{A}_2^1)$ is too.

The picture 3.2 depicts the situation we have so far, with $\mathcal{A}^1$ coding the well-founded part of $\mathcal{N}$, L_α for some $\text{wfo}(\mathcal{N}) \coloneqq \alpha$.

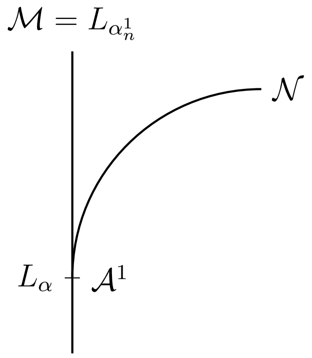

Figure 3.2: A typical situation in the game of KP^2_n

We start our stalking of the ill-founded structure by sounding the unique respective Σ_1-definable subsets of each model. To this aim, we define the following set.

$$W^1_{\mathcal{M}_{\mathrm{I}},1} = \quad \{\beta \in Ord^{\mathcal{M}_{\mathrm{I}}} \mid \exists(x_1, x_2) \in \mathcal{A}^1, \phi \in \Delta_0, [(\exists z \in L^{\mathcal{M}_{\mathrm{I}}}_\beta \; \mathcal{M}_{\mathrm{I}} \models \phi(z, x_1)) \wedge (\mathcal{M}_{\mathrm{II}} \models \neg\exists y \phi(y, x_2))]\}.$$

$W^1_{\mathcal{M}_{\mathrm{II}},1}$ is defined *mutatis mutandis*. The next conditions are then

$$(C_{\mathrm{I}}2): \quad W^1_{\mathcal{M}_{\mathrm{I}},1} \text{ has a least element or is empty.}$$
$$(C_{\mathrm{II}}2): \quad W^1_{\mathcal{M}_{\mathrm{II}},1} \text{ has a least element or is empty.}$$

This condition says that $\exists\beta(\beta \in W^1_{\mathcal{M}_{::}} \wedge \forall\gamma < \beta(\gamma \notin C^1_{\mathcal{M}_{::}}))$. Since $W^1_{\mathcal{M}_{::},1}$ is Σ^0_3, it is a Σ^0_4 condition. Now comes a key fact that will start our inductive search for the ill-founded model.

Lemma 3.2.6. *Suppose conditions $C_{::}i$ to be satisfied for $i = 0, 1, 2$. Then there is a $\beta \in Ord^{\mathcal{N}} \setminus \mathcal{A}^1$ such that $\mathcal{A}^1 \preceq_1 L^{\mathcal{N}}_\beta$.*

Proof. Suppose for a contradiction that for every $\gamma \in Ord^{\mathcal{N}} \setminus \mathcal{A}^1$, there is a Σ_1 formula with parameters in $\mathcal{A}^1$ true in L_γ but not in $\mathcal{A}^1$. By hypothesis, $W^1_{\mathcal{N},1}$ has a least element δ. By definition of $W^1_{\mathcal{N},1}$, we have $\delta \notin \mathcal{A}^1$. Since from this point we suppose $\mathcal{N}$ to be ill-founded let

$$\delta > \gamma_0 > \gamma_1 > \gamma_2 > \cdots$$

be a descending sequence in $Ord^{\mathcal{N}} \setminus \mathcal{A}^1$, converging down to the cut $(Ord^{\mathcal{A}^1}, Ord^{\mathcal{N}} \setminus \mathcal{A}^\infty)$. By our absurd assumption, for each i, there is a Δ_0 formula ϕ_i with parameters in $\mathcal{A}^1$ and a $<_L$-least witness $z_i \in L_{\gamma_i}$ such that

$$\mathcal{N} \models \phi_i(z_i) \quad \text{but} \quad \mathcal{A}^1 \models \neg\exists y \phi_i(y).$$

By thinning out our sequence if necessary, we may assume that $z_i \notin L_{\gamma_i+1}$ so that $z_i : i < \omega$ is an $<_L^{\mathcal{N}}$-descending sequence. Since we assumed by our absurd hypothesis that δ was the least element of $W^1_{\mathcal{N},1}$, for all i,

$$\mathcal{M} \models \exists y \phi_i(y).$$

Let y_i be the $<_L^{\mathcal{M}}$-least such witnesses. Since $\mathcal{M}$ is well-founded, the sequence $\{y_i : i < \omega\}$ cannot be an $<_L^{\mathcal{M}}$-descending sequence. So there exist two index $i < j$ such that

$$z_j <_L^{\mathcal{N}} z_i \quad \text{but} \quad y_i <_L^{\mathcal{M}} y_j.$$

Therefore, $L_{\gamma_{j+1}}$ is a witness in L_{γ_j} for the Δ_0 formula

$$\psi(x) \equiv \exists (z \in x)\ \phi_j(z) \wedge \forall (z \in x)\ \neg \phi_i(z),$$

that is true in $\mathcal{N}$ but not in $\mathcal{M}$ where we have $y_i <_L^{\mathcal{M}} y_j$. This however shows that γ_{j+1} is an element of $W^1_{\mathcal{N},1}$, a contradiction. □

The goal now is to define the remaining conditions such that if all the conditions $C_{::}i$ for $i = 0, 1, \ldots k$ hold, then $\mathcal{A}^1$ codes an initial segment satisfying KP^2_k. However, by definition of T^1_n, such a (strict) initial segment can't be a model of KP^2_n and thus one of the conditions we are about to define is doomed to fail. We will prove this by induction and to perform this we need the following induction hypothesis. We want $\wedge_{i=0}^k C_{::}i$ to imply the existence of β_1 and β_2 such that

$$\begin{aligned}(\star_k)(\beta_1, \beta_2): \qquad & \beta_1 \in Ord^{\mathcal{M}_{\mathrm{I}}} \setminus \mathcal{A}^1_{\mathrm{I}} \wedge \mathcal{M}_{\mathrm{I}} \models L_{\beta_1} \text{ satisfies } \mathsf{KP}^2_{k-1} \wedge \\ & \beta_2 \in Ord^{\mathcal{M}_{\mathrm{II}}} \setminus \mathcal{A}^1_{\mathrm{II}} \wedge \mathcal{M}_{\mathrm{II}} \models L_{\beta_2} \text{ satisfies } \mathsf{KP}^2_{k-1} \wedge \\ & L^{\mathcal{M}_{\mathrm{I}}}_{\beta_1} \equiv_{k,\mathcal{A}} L^{\mathcal{M}_{\mathrm{II}}}_{\beta_2},\end{aligned}$$

where $\equiv_{k,\mathcal{A}^1}$ is written for Σ_k elementary equivalence, with parameters from $\mathcal{A}^1$ and $z \in \mathcal{A}^1_I \leftrightarrow \exists w\ (z, w) \in \mathcal{A}^1_I$, a Σ^0_3 property. By lemma 3.2.6, we know that so far, such a pair of ordinals of the respective models satisfying $(\star_1)$ exists. Indeed, we can take $\beta_1 = \alpha \in \mathcal{M}$ (from L_α) and $\beta_2 = \beta \in \mathcal{N}$ (given by the lemma). For the sake of definiteness, we will always assume that player I is playing the well-founded model. What is more, the above property $(\star_k)$ is Π^0_3, the most complex property being "$\beta_1 \notin \mathcal{A}^1_{\mathrm{I}}$".

Definition 3.2.7 (S_k formulae)**.** *A formula of second-order arithmetic is said to be S_k if it is a Boolean combination of formulae of the form $(\forall x \in z)\ \psi(x, \bar{y})$ where $\bar{y}$ are free variables and ψ is Σ_k.*

Lemma 3.2.8. *If $L^{\mathcal{M}_{\mathrm{I}}}_{\beta_1} \equiv_{k,\mathcal{A}^1} L^{\mathcal{M}_{\mathrm{II}}}_{\beta_2}$, then $L^{\mathcal{M}_{\mathrm{I}}}_{\beta_1}$ and $L^{\mathcal{M}_{\mathrm{II}}}_{\beta_2}$ satisfies the same S_k-sentences with parameters from $\mathcal{A}^1$ substituted for the free variables z and $\bar{y}$.*

Proof. This is because $\mathcal{A}^1$ is transitive since then, given a formula of the form $(\forall x \in z)\ \psi(x,\bar{y})$ and $z,\bar{y} \in \mathcal{A}^1$ for any $x \in z$, the sentence $\psi(x,\bar{y})$ is Σ_k with parameters from $\mathcal{A}^1$. Then by definition of "$\models$" the claim follows easily. □

We are ready to move on and dive into the definitions of the generalized versions of the sets $W^1_{\mathcal{M}_{::},1}$. This time we search for non-standard Σ_k-definable subsets of each model $(1 < k)$. Once again, either they are a lot of such elements, witnessing an infinite descending sequence and thus betraying the identity of the ill-founded model, or their rarity implies the existence of ordinals satisfying $(\star_k)$.

$$\begin{aligned} W^{1;\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{I}},k} = \quad & \{\beta \in \beta_1 \mid \exists (x_1,x_2) \in \mathcal{A}^1, \phi \in S_{k-1}, [(\exists z \in L^{\mathcal{M}_{\mathrm{I}}}_{\beta}\ L^{\mathcal{M}_{\mathrm{I}}}_{\beta_1} \models \phi(z,x_1)) \wedge \\ & (L^{\mathcal{M}_{\mathrm{II}}}_{\beta_1} \models \neg \exists y \phi(y,x_2))]\}, \end{aligned}$$

and $W^{1;\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k}$ is defined *mutatis mutandis*. As before, these sets are Σ^0_3. Now let us define the remaining conditions involved in determining the winner of the game (as before we treat indexes $k > 1$).

$$\begin{aligned} (C_{\mathrm{I}}(1+k)): \quad & \text{There exist } \beta_1,\beta_2 \text{ such that } (\star_{k-1})(\beta_1,\beta_2) \\ & \wedge W^{1;\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{I}},k} \text{ has a least element or is empty.} \\ (C_{\mathrm{II}}(1+k)): \quad & \text{There exist } \beta_1,\beta_2 \text{ such that } (\star_{k-1})(\beta_1,\beta_2) \\ & \wedge W^{1;\beta_1,\beta_2}_{\mathcal{M}_{\mathrm{II}},k} \text{ has a least element or is empty.} \end{aligned}$$

By inspection of the alternation of quantifiers highlighted in the preceding definitions, these last conditions are Σ^0_4.

Lemma 3.2.9. *Suppose that β_1,β_2 satisfy $(\star_k)$. Then*

1. $L_\alpha \preceq_k L_{\beta_1}$ *and* $\mathcal{A}^1 \preceq_k L_{\beta_2}$;
2. $L_\alpha \models \mathsf{KP}^2_{k+1}$;
3. *There exists a descending sequence of $\mathcal{N}$-ordinals γ converging down to $Ord^{\mathcal{A}^1}$ such that $L_\gamma \preceq_k L_{\beta_2}$.*

Proof. First, we claim that α is not Σ_k definable in L_{β_1}, with parameters from L_α. Since $\alpha \in \mathcal{M} \models T^1_n$, it follows that α is not n-admissible and by lemma 2.2.9, every $\beta \in \mathcal{M}$ is of cardinality at most $\mathcal{P}(\omega)$. Thus there is a Σ^n definable map on L_α from $\mathcal{P}(\omega)$ onto α, eventually with parameters. This defines in L_α a Σ_n well ordering of $\mathcal{P}(\omega)$ of order type α. This ordering cannot belong to $\mathcal{N}$ as it would define its well-ordered part. By our absurd hypothesis, in L_{β_1}, we have a Σ_k definition of this ordering using the Σ_k definition of α and bounded quantification over L_α. However then, since $L^{\mathcal{M}_{\mathrm{I}}}_{\beta_1} \equiv_{k,\mathcal{A}^1} L^{\mathcal{M}_{\mathrm{II}}}_{\beta_2}$, this ordering is now definable in L_{β_2} and hence belongs to $\mathcal{N}$, a contradiction.

For point 1 it is of course sufficient to show $L_\alpha \preceq_k L_{\beta_1}$ since the other is Σ_k elementary equivalent to it, over $\mathcal{A}^1$. Since β_1 is $(k-1)$-admissible, from lemma 2.2.6 we know that L_{β_1} has a parameterless Σ_k Skolem function. Let thus be H, the Σ_k-Skolem hull of L_α in L_{β_1}. We show that $H = L_\alpha$, which will prove our claim. Suppose otherwise towards a contradiction and consider L_γ, the Mostowski collapse of H (see theorem 2.1.6), with $\alpha < \gamma \leq \beta_1$. Let α' be the ordinal of H being sent to $\alpha \in L_\gamma$ by the collapse. By construction of H, we would have a Σ_k definition of α in L_γ, with still parameters from L_α, since the collapse is the identity over L_α. However since

$$L_\gamma^{\mathcal{M}} \equiv_{k,\mathcal{A}^1} H \equiv_{k,\mathcal{A}^1} L_{\beta_1}^{\mathcal{M}},$$

α would be Σ_k definable in $L_{\beta_1}^{\mathcal{M}}$, a contradiction.

Next concerning point 2, let us, as usual, suppose that our claim is false and thus α is not $k+1$-admissible and so there is a Π_k definable map on L_α from $\mathcal{P}(\omega)$ onto α. Since $\mathcal{A} \preceq_k L_{\beta_2}$, as for our first observation, we would have that $\mathcal{N}$ would be able to define its well-founded part and a contradiction would occur.

Finally about point 3, we use theorem 2.1.8 to get from the freshly proved $(k+1)$-admissibility of α, the existence of an unbounded infinity of $\gamma < \alpha$ such that $L_\gamma \preceq_k L_\alpha \preceq_k L_{\beta_2}$. The set of the $\gamma < \beta_2$ such that $L_\gamma \preceq_k L_{\beta_2}$ is on the other hand definable in $\mathcal{N}$. So now if for some $\delta \in Ord^{\mathcal{N}} \setminus \mathcal{A}^1$ this set had supremum α, then α would be definable in $\mathcal{N}$ and we know it is not. So for every $\delta \in Ord^{\mathcal{N}} \setminus \mathcal{A}^1$, there exists $\delta > \gamma \in Ord^{\mathcal{N}} \setminus \mathcal{A}^1$ such that $L_\delta \preceq_k L_{\beta_2}$. □

Lemma 3.2.10. *If there is a play of our game such that, for all $i \leq 1+k$, the resulting real satisfy all the conditions $(C_{\mathrm{I}}i)$ and $(C_{\mathrm{II}}i)$ for all $0 \leq i \leq 1+k$, then there are β_1 and β_2 satisfying $(\star_k)$.*

Proof. Let us prove our claim by induction, lemma 3.2.6 giving us the base step. Assume there exist some fixed β_1 and β_2 satisfying $\star_{k-1}$.

Firstly we claim that no ordinal $\delta \in W_{\mathcal{N},k}^{1;\beta_1,\beta_2}$ is in $\mathcal{A}^1$. Let $\delta \in \mathcal{A}^1$, any S_{k-1} formula $\forall (x \in z)\phi(z, \bar{y})$ and $z_2, \bar{y}_2 \in L_\delta^{\mathcal{N}} \subseteq A$ such that $L_{\beta_2}^{\mathcal{N}} \models \forall (x \in z_2)\phi(z_2, \bar{y}_2)$. By induction hypothesis, it follows that $L_{\beta_2}^{\mathcal{N}} \models \forall (x \in z_1)\phi(z_1, \bar{y}_1)$ too, with z_1 and $\bar{y}_1$ the images of z_2 and $\bar{y}_2$ in $\mathcal{M}$ (via the isomorphism $\mathcal{A}^1$). Thus $\delta \notin W_{\mathcal{N},k}^{1;\beta_1,\beta_2}$.

Now, by hypothesis, $W_{\mathcal{N},k}^{1;\beta_1,\beta_2}$ has a least element δ, necessarily not in $\mathcal{A}^1$. Also by clause 3 of lemma 3.2.9, there is a descending sequence

$$\delta > \gamma_0 > \gamma_1 > \gamma_2 > \ldots$$

in $Ord^{\mathcal{N}}$ converging down to $\alpha = Ord^{\mathcal{N}}$, such that, for each $i < \omega$, $L_{\gamma_i}^{\mathcal{N}} \preceq_{k-1} L_{\beta_2}^{\mathcal{N}}$. Now we argue exactly like lemma 3.2.6 (where we had $k = 1$ and $\preceq_0$ is absoluteness for Δ_0

formula, which follows from the transitivity of the structures) to get that for some $i < \omega$,

$$L_\alpha \preceq_k L^{\mathcal{N}}_{\gamma_i} \preceq_{k-1} L^{\mathcal{N}}_{\beta_2}.$$

Finally we conclude by lemma 2.2.8 that $L^{\mathcal{N}}_{\gamma_i}$ is $(k-1)$-admissible as $L^{\mathcal{N}}_{\beta_2}$ is $(k-2)$-admissible by induction hypothesis while α is (even) Σ_k admissible by lemma 3.2.9, so that $\star_k(\alpha, \gamma_i)$, as required. □

Before putting our game under its final presentation (a $(\Pi^0_4)_n$ game), we prove that the $(\Pi^0_4)_{2n+2}$ game G'^1_n, whose payoff set is described by the sequence

$$(C_{\mathrm{II}}0), (C_{\mathrm{I}}0), (C_{\mathrm{II}}1), (C_{\mathrm{I}}1), \ldots, (C_{\mathrm{II}}(1+k)), (C_{\mathrm{I}}(1+k)), \ldots, (C_{\mathrm{II}}n), (C_{\mathrm{I}}n),$$

satisfy a behaviour analogous to the one defined in the proof of theorem 3.2.3.

Lemma 3.2.11. *The game G'^1_n we just defined satisfy:*

1. *If player* I *plays the theory of $L_{\alpha^1_n}$, she wins;*
2. *If player* I *does not play the theory of $L_{\alpha^1_n}$ but player* II *does, then player* II *wins.*

Proof. Taking the model produced by the theory of $L_{\alpha^1_n}$, since $Ord^{\mathcal{A}} = \alpha \in \mathcal{M} \models Th(L_{\alpha^1_n})$, α cannot be n-admissible (we can suppose $C_{\mathrm{II}}0, 1$ didn't failed already). So by lemmas 3.2.10 and 3.2.9, there is a $k < n$ such that either $C_{\mathrm{I}}(1+k)$ or $C_{\mathrm{II}}(1+k)$ fails. Suppose $C_{\mathrm{II}}(1+k)$ is the first condition to fail and that I wins the game. Since $\forall i < 1+k$, all the conditions $C_{::}(i)$ are satisfied, this failure means that $\mathcal{M}_{\mathrm{II}}$ is ill-founded. An analogous argument works when $C_{\mathrm{I}}(1+k)$ is the first condition to fail and II thus wins the game. □

Now we will modify the game, in order to still satisfy lemma 3.2.11, but with a $(\Pi^0_4)_{n-1}$ game that we will call G^1_n. The first real restriction is that we need to check both $(C_{\mathrm{I}}0), (C_{\mathrm{I}}1)$ and $(C_{\mathrm{II}}0), (C_{\mathrm{II}}1)$ before to begin with the rest of the conditions if we want our game to work as intended. The other one is that we must place the conditions $(C_{::}(k+2))$ after both conditions $(C_{::}(k+1))$, since we are only interested in the first condition to fail, it means that the preceding ones hold; what we need for the $(C_{::}(k+1))$ to work as intended. Otherwise, a condition $(C_{::}(k+2))$ could fail for the player playing the well-founded model, just because the condition $(C_{::}(k+1))$ did not hold for the player playing the ill-founded model, but we did not check it earlier. On the other hand, since only the player playing $\mathcal{N}$ can lose by such a condition, the order of $(C_I(k+2))$ and $C_{\mathrm{II}}(k+2)$ does not matter.

From these observations, the $(\Pi^0_4)_n$ game G^1_n is defined by the condition depicted on figure 3.3.

Even:	$(C_{II}0,1)$		$(C_{II}2,3)$	$\dots$	$(C_{II}n)$
Odd:		$(C_I0,1,2)$		$(C_I3,4)$	

Figure 3.3: The game G^1_n for n even.

Proof of theorem 3.2.4. The proof is the same as the one of theorem 3.2.3, taking the game defined above, G^1_n that satisfies lemma 3.2.11. □

We see that now that we have overcome the first "uncountable" case, we can generalize the proof by replacing the emphasised index "1" with "$1 \leq i - 1 < \omega$". The generalisation of $\mathcal{A}^1$ need to be somehow cautious since the number of cases will increase when determining the common well-founded part but we will get Σ^0_{3+i} conditions, because of the higher imbrication of quantifiers coming from the higher order type of the objects, and be dealing with theories including KP^i_n. This allows us to state the following theorem.

Theorem 3.2.12. *For all $2 \leq i, n < \omega$,*

$$\mathsf{KP}^i_n := \mathsf{KP} + \text{“}\mathcal{P}^{i-1}(\omega) \text{ exists”} + \Delta_{n-1} \text{ COLLECTION} + \Sigma_{n-1} \text{ SEPARATION} \nvdash (\Pi^0_{2+i})_n\text{-}\mathsf{Det}.$$

Furthermore, by theorem 2.2.7, we also get for free from the proof that even reinforcing our theory with Δ_n Separation is not sufficient to prove $(\Pi^0_{2+i})_n$-Det.

Conclusion

At the beginning of this thesis, we were wondering what was the strength of determinacy axioms in the context of Gale-Stewart games with Borel payoff $(\Pi^0_n)_m$ $(3 \leq n, m-2 < \omega)$. It was already known from the result of Martin [35] that this was a list of theorems that were unsolvable without sufficient iteration of the power set axioms at disposal because of how quickly the complexity of the winning strategies of such games grow. The paper of Montalbán and Shore [38] was also telling us that the hierarchy of differences $(\Pi^0_3)_m$ was growing very quickly in terms of logical requirements for the existence of winning strategies in games of such payoff set, from Π^1_2 comprehension to the very top of the subsystems of Z_2. Along the way, this gives us many other examples of Gödel's incompleteness theorem, way before the very powerful theory ZFC.

From the first preliminary chapter of the present thesis to the end, we introduced determinacy and reverse mathematics and presented, in the absence of precise reverse mathematics reversals, a narrow gap for $(\Pi^0_3)_m$-Det giving us a taste of the reverse mathematics method. We also presented a very central method in the field, of interpretations back and forth between second-order arithmetic and some fragments of set theory, starting from ATR_0. This allowed us to enrich a lot our analysis of determinacy axioms for these payoff sets. Not only that but by implementing set theory tools in second-order arithmetic, especially constructibility and admissible sets of Barwise [3] and Devlin [10], we developed crucial conservation results and better bounds (the one presented in [38]) for the determinacy of differences. We finally exposed the way the second theorem of Montalbán and Shore could be generalised with the same theories of Kripke-Platek as the ones modelling subsystems of second-order arithmetic but with the existence of higher type objects.

Determinacy is a very wild subject for reverse mathematics however, as it can be strengthened or weakened in various ways like modifying the length of the trees in the games, the space in which we play the moves or the complexity of the payoff sets. Before all, the next step of this thesis is to generalise the first theorem of [38]. Beyond these theorems of Montalbán and Shore, many questions remain, such as the precise strength of $(\Pi^0_3)_m$-Det and others, as Hachtman proves for the levels of the Borel Hierarchy of the form $\Sigma^0_{1+\alpha+3}$ in [17]. Other directions are studying determinacy in third-order arithmetic, since games with moves from the real numbers can be described within it, as it is discussed in [19], or even determinacy as independent questions from ZFC and their links with large cardinals, sharps, mice, etc, which we can learn about in [25, 35].

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