A study about Hilbert's fifth problem for local groups

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Under the supervision of Françoise Point

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Introduction

In 1900, the mathematician D.Hilbert, on his famous list of problems, asked whether every continuous group of transformations of a finite dimensional real or complex space is a Lie group. A couple of decades later, O.Schreier defined topological groups, abstracting the implicit idea, already present in S.Lie's work, of continuous groups of infinitesimal transformations. A new formulation of Hilbert's question then arose: which topological conditions on a topological group will ensure that it has a structure which makes it a Lie group? It was answered in 1952 by A.Gleason, D.Montgomery and L.Zippin: a topological group is a Lie group if and only if it is locally Euclidean, namely if and only if it is locally compact and has no small subgroups (i.e. there is a neighborhood of the identity containing no non-trivial subgroup). These two answers are now considered as the most common versions of Hilbert's fifth problem (H5).

As the structure of a Lie group is local, a local version of Hilbert's fifth problem (local H5) came naturally: is every locally Euclidean local group locally isomorphic to a Lie group? A local group being, roughly speaking, a Hausdorff topological space with an identity element, and continuous inverse and multiplication maps that are not defined everywhere, and in which an associativity law, called local, holds for products of three elements. However, associativity in local groups does not necessarily hold for products of more than four elements, unless these elements are sufficiently close to the identity: for instance, given elements x, y, u, v of a local group G, products such as $(x(yu))v$ and $(xy)(uv)$ may be defined but not equal. We will give precise definitions of local groups and of a global associativity law for local groups in Chapter 1.

In 1957, R.Jacoby proposed an affirmative answer to local H5 in [**10**], but one of the theorems on which the solution relied was false: as noticed by C.Plaut in [**17**], and further developped by P.Olver in [**14**], it did not make a distinction between local and global associativity.

Meanwhile, in 1934, T.Skolem constructed the first nonstandard models of Arithmetic, using model-theoretic techniques. In 1961, a similar construction was used by A.Robinson (see for example [**19**]) to construct nonstandard real numbers (or more generally nonstandard topological spaces), in which there are infinitesimal and infinite elements. This allowed to formalize concepts from the 17th century, that were not used any longer because of their lack of precision. As H5 involves limits and asymptotic behaviours, nonstandard analysis appeared to be a quite appropriate framework: in 1990, J.Hirschfeld used it in [**7**], in order to give a simplified proof of H5.

I.Goldbring, in [**5**], in 2010, instead of fixing Jacoby's proof, chose to adapt Hirschfeld's proof to local H5. Working with local groups, one is brought to consider classes of restrictions of them to given neighborhoods of the identity. Nonstandard methods allow to consider a set of infinitesimal elements instead, which is actually a genuine group. Hence, most of the proof of H5 goes through in the local case. I.Goldbring consequently follows Hirschfeld's proof, by first showing that every locally compact local group with no small subgroups (NSS) has a restriction which is a local Lie group, and then by showing that every locally Euclidean local group is NSS.

Our work consists in first developping the nonstandard setting used in Goldbring's proof of local H5, and then in giving a detailed account of his proof that the set $L(G)$ of equivalence classes of some local 1-parameter subgroups can be equipped with an abelian group law. From this, it is not too difficult to show that $L(G)$ is a locally compact real vector space, hence a finite dimensional real vector space, by a theorem of Riesz.

In Chapter 1, we define local groups and state local H5, with a special care given to the two different notions of associativity, and how they are linked to local H5.

In Chapter 2, we study the model-theoretic constructions underlying a nonstandard setting with [**3**] C.C.Chang et H.J.Keisler *Model Theory*, taking the ultraproduct approach. Some combinatorial properties of ultrafilters allow to construct ultraproducts in which, for some cardinal κ, every intersection of κ definable sets is nonempty provided that every finite intersection of them is nonempty. Such a structure is called κ^+-saturated.

In Chapter 3, starting from a local group G and from a base $\mathcal{B}$ for its topology, we expand the group language adding a unary predicate for each element of $\mathcal{B}$. We then consider a $|\mathcal{B}|^+$-saturated ultrapower G^* of G. The infinitesimal elements in G^* (i.e. the elements lying in all neighborhoods of the identity) form a group μ, that is used to construct a kind of 'dictionnary' expressing properties of G within G^*.

In Chapters 4 and 5, following I.Goldbring, we see how to construct local 1-parameter subgroups (local 1-ps) from a certain kind of infinitesimals: an idea underlying the proofs of Goldbring and Hirschfeld is, roughly speaking, kind of cutting the group μ into 'slices', with the help of normal subgroups of μ corresponding to different ways the powers of these infinitesimal elements can grow. By taking a quotient, it is possible to get an intermediate 'slice' corresponding to infinitesimals the powers of which grow neither too fast nor too slow.

In Chapter 6, we see the local adaptation of some important technical lemmas of H5, namely the Gleason-Yamabe Lemmas. To each compact neighborhood of the identity with certain properties, a continuous function is associated. Then, linking G to the set C of continuous functions from G to $\mathbb{R}$ with compact support, the Haar measure is defined. Then properties of $||af - f||$, where a belongs to a neighborhood of the identity in G and f is a particular function that belongs to C, are shown in order to get information about a.

In the last Chapter, we study a nonstandard version of the Gleason-Yamabe Lemmas, relative to a set $Q \subseteq \mu$. Then we prove that it is possible to endow $L(G)$ with the structure of an abelian group by showing that a quotient (corresponding to the 'good slice' of infinitesimals previously mentioned) is abelian. This is were we end our partial exposition of local H5. Nevertheless, we try to give a sketch of the remaining points to be dealt with in order to complete the proof.

CHAPTER 1

The local Hilbert's fifth problem

Section mainly based on [**5**], [**14**], but also on [**22**] and [**13**] and [**23**].

1. Preliminaries

Let (X, τ) be a topological space. We first recall some separation axioms; proofs of the propositions and remarks can be found in [**1**] or in [**23**].

DEFINITION 1.1. *Let (X, τ) be a topological space.*

- *(**$\mathbf{T_0}$**) (X, τ) is T_0 iff given two distinct points $x, y \in X$, there is a neighborhood of one of them not containing the other.*
- *(**$\mathbf{T_1}$**) (X, τ) is T_1 iff given two distinct points $x, y \in X$, there is a neighborhood of each not containing the other.*
- *(**$\mathbf{T_2}$**) (X, τ) is T_2 (or equivalently Hausdorff or separated) iff given two distinct points $x, y \in X$, there is a neighborhood V_x of x and a neighbourhood V_y of y such that $V_x \cap V_y = \emptyset$.*
- *(**regular**) (X, τ) is regular iff whenever A is closed in X and $x \notin A$, then there are disjoint open sets U_x and V_A such that $x \in U_x$ and $A \subseteq V_A$.*
- *(**completely regular**) (X, τ) is completely regular iff whenever A is closed in X and $x \notin A$, then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.*
- *(**normal**) (X, τ) is a normal space iff whenever A and B are closed disjoint subsets of X, then there are disjoint open sets O_A and O_B such that $A \subseteq O_A$ and $B \subseteq O_B$.*
- *(**$\mathbf{T_3}$**) (X, τ) is T_3 iff it is T_1 and regular.*
- *(**$\mathbf{T_{3^{1/2}}}$**) (X, τ) is $T_{3^{1/2}}$ (or equivalently **Tychonoff**) iff it is T_1 and completely regular.*
- *(**$\mathbf{T_4}$**) (X, τ) is T_4 iff it is T_1 and normal.*

PROPOSITION 1.2. *(X, τ) is regular iff for all $x \in X$ the set of neighborhoods of x has a base of closed sets iff whenever U is open in X and $x \in U$, there is an open set V such that $x \in V$ and the closure of V is included in U.*

LEMMA 1.3 (**Urysohn Lemma**). *Let A and B be closed disjoint subsets of a normal topological space X. Then there exists a continuous function $f : X \to [0; 1] \subseteq \mathbb{R}$ such that for all $x \in A$, $f(x) = 0$ and for all $x \in B$, $f(x) = 1$.*

We have that $T_4 \Rightarrow T_{3^{1/2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

A compact topological space is normal. A topological space which is locally compact and T_2 is Tychonoff and consequently completely regular.

DEFINITION 1.4 (**separable**). *(X, τ) is separable if there is a subset $A \subseteq X$, such that A is finite or countable, and such that the closure of A is X.*

Note that if X is separable, a subspace of X is not necessarily separable. If there exists a countable basis for the topology τ, X is then separable. The other implication is not true, except if one adds the assumption for X to be also metrizable.

DEFINITION 1.5. *$G := (G, 1, ., ^{-1})$ is a **topological group** if it is a group and if it is equipped with a topology $\tau \subseteq \mathcal{P}(G)$ for which the group operations $., ^{-1}$ are continuous.*

Any group can be made into a topological group by putting on it the discrete or the indiscrete topology.

PROPOSITION 1.6. *Topological groups are completely regular.*

PROOF. See [**13**] □

In the literature topological groups are generally supposed to be T_2 ; in fact it is easy to see that when they are T_0 they become automatically T_2 (and thus even Tychonoff because they are completely regular). It is also possible to reduce the study of topological groups to the study of T_2 topological groups, by pointing out that :

- A subgroup H of a topological group is itself topological when given the induced topology. The closure of H is still a subgroup. If H is normal in G, the closure of H is also normal.
- If H is a subgroup of G, the set of left cosets G/H (i.e. the quotient of G by the equivalence relation $x^{-1}y \in H$) is a topological space when given the quotient topology.
  If H is normal, then G/H with the quotient topology is a topological group.
  If H is closed, then G/H is T_2. If not, G/H will not be T_0.
- G is T_2 iff $\{1\}$ is closed in G.
  As the closure of $\{1\}$ is always a closed normal subgroup H of G,the study of G can be reduced to the one of a T_2 topological group, namely G/H.

We now state some basic properties about the topological structure of a topological group. Proofs can be found in [**1**] or [**13**]. Anyway, most of these proofs will be seen later on, in the more general case of local groups.

Let $g \in G$ and V be an open set containing 1. Then gV and Vg are open sets containing g.

This can also be expressed by saying that the left and right translations from G to G, namely L_g and R_g, are continuous (in fact they are homeomorphisms).
The topology on G is thus determined by the topology near the identity, or, equivalently, by the neighborhoods of 1.
Note also that an open subgroup H of G is closed (as G/H is a union of cosets of H, since the set of orbits of the action of H on G is a partition of G) ; hence if G is compact every open subgroup has finite index ; and that if V is an open set of G, then so is $V^{-1} := \{v^{-1} : v \in V\}$. A set V is called symmetric if $V = V^{-1}$.

PROPOSITION 1.7. *Let V be an open neighborhood of 1. Then there exists a symmetric open neighborhood W of 1 such that $W.W \subseteq V$.*

PROPOSITION 1.8. *Let G be a topological group, 1 its identity element. Let U be a neighborhood of 1. If G is connected, the subgroup generated by U is G.*

DEFINITION 1.9. *Let M be a topological space.*

- *A local map of dimension n for M is a pair (U, ϕ) where U is an open set of M and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism.*
- *An atlas $\mathcal{A}$ of dimension n for M is a collection*
$$\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$$
where I is a set, and where the pairs (U_i, ϕ_i) are local maps of dimension n for M, such that $M = \bigcup_{i \in I} U_i$.
- *M is said to be **locally euclidean** if it is equipped with an atlas of dimension n for some n.*
- *An atlas $\mathcal{A}$ is smooth (or equivalently C^ω) if for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the map*
$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subseteq \mathbb{R}^n \to \phi_i(U_i \cap U_j) \subseteq \mathbb{R}^n$$
is smooth (or equivalently differentiable or C^ω).
- *M is said to be a **smooth manifold (of dimension** n**)** if M is T_2, if its topology has a countable basis, and if it is equipped with a maximal smooth atlas (of dimension n).*

Note that if M is locally euclidean, it is locally compact.
If M is a smooth manifold with atlas $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$, $M \times M$ is also a smooth manifold with atlas $\mathcal{B} := \{(U_i \times U_j, \phi_i \times \phi_j) : (i, j) \in I \times I\}$.

DEFINITION 1.10. *Let M, N be smooth manifolds, and $f : M \to N$ an application.*

- *Let $p \in M$. We say that f is smooth at p if there exists a local map (U, ϕ) such that $p \in U$ and a map (V, ψ) such that $f(U) \subseteq V$ and $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth at $\phi(p)$.*
- *We say that f is smooth if f is smooth at every $p \in M$.*

Definition 1.11. *A group G is a* ***Lie group*** *if it is also a smooth manifold and if the group operations $. : G \times G \to G$ and $^{-1} : G \to G$ are smooth.*

For instance, $\mathbb{R}^n$ equipped with addition is a Lie group, as well as $GL(n, \mathbb{R})$, the set of $n \times n$ matrix with real coefficients and determinant different from 0, equipped with multiplication.

Lie groups are T_2 as smooth manifolds, so we are interested in T_2, or T_0 topological groups, and in T_2 topological spaces.

2. Local groups

Let G be a topological group, and let V be a neighborhood of the identity in G, to which we look as a space with the induced topology. Let $x, y \in V$. The product xy and the inverse element x^{-1} do not necessarily lie in V. This kind of structure is in fact a generalisation of the one of topological groups. We define it below:

Definition 1.12. *A* ***local group*** *is a 4-tuple $(G, 1, \iota, p)$ where G is a T_2 topological space with a distinguished element $1 \in G$, and $\iota : \Lambda \to G$ (the inversion map) and $p : \Omega \to G$ (the product map) are continuous functions with open set $\Lambda \subseteq G$ and open set $\Omega \subseteq G \times G$, such that $1 \in \Lambda$, $\{1\} \times G \subseteq \Omega$, $G \times \{1\} \subseteq \Omega$, and for all $x, y, z \in G$:*

(i) $p(1, x) = p(x, 1) = x$

(ii) *(**local inverse**) if $x \in \Lambda$, then $(x, \iota(x)) \in \Omega$, $(\iota(x), x) \in \Omega$ and*

$$p(x, \iota(x)) = p(\iota(x), x) = 1 \tag{2.1}$$

(iii) *(**local associativity**) if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then*

$$p(p(x, y), z) = p(x, p(y, z)) \tag{2.2}$$

We will often abuse notation and identify G and $(G, 1, \iota, p)$.

Definition 1.13. *Let $G = (G, 1, \iota, p)$ and $G' = (G', 1', \iota', p')$ be local groups with domain(ι)= Λ, domain(p)= Ω, domain(ι')= Λ', domain(p')= Ω'. A* ***morphism*** *from G to G' is a continuous function $f : G \to G'$ such that*

(i) $f(1) = 1'$, $f(\Lambda) \subseteq \Lambda'$ *and* $(f \times f)(\Omega) \subseteq \Omega'$

(ii) $f(\iota(x)) = \iota'(f(x))$ *for* $x \in \Lambda$, *and*

(iii) $f(p(x, y)) = p'(f(x), f(y))$ *for* $(x, y) \in \Omega$

Lemma 1.14. *(**homogeneity**) Let G be a local group such that $\Lambda = G$.*

(i) *For any $g \in G$, there are open neighborhoods V and W of 1 and g respectively such that $\{g\} \times V \subseteq \Omega$, $gV \subseteq W$, $\{\iota(g)\} \times W \subseteq \Omega$, $\iota(g)W \subseteq V$, and the maps*

$$v \mapsto p(g, v) : V \to W \text{ and } w \mapsto p(\iota(g), w) : W \to V$$

are each other inverses (and hence homeomorphisms) *(ii) G is locally compact iff there is a compact neighborhood of* 1*.*

The proof of this lemma is not straightforward as it is the case for topological groups.

Proof. $(i) \Rightarrow (ii)$ is clear, so we only need to prove (i).
Let $g \in G$ and $\Omega_g := \{h \in G : (g,h) \in \Omega\}$. Ω_g is an open subset of G, as a projection of the open subset Ω of $G \times G$. The map

$$L_g : \begin{array}{ccc} \Omega_g & \to & G \\ h & \mapsto & p(g,h) \end{array}$$

is continuous : let O be an open set containing $p(g,h)$. As $(g,h) \in \Omega$ which is open in the product topology, there are $O_1, O_2 \subseteq G$ open and such that $(g,h) \in O_1 \times O_2 \subseteq \Omega$. Moreover $O_2 \subseteq L_g^{-1}(O) := \{h \in G : p(g,h) \in O\}$, hence $L_g^{-1}(O)$ is open, so L_g is well continuous.
Now let $V := L_g^{-1}(\Omega_{\iota(g)})$. Note that V is included in Ω_g by definition of the map L_g. The set V is open by continuity of L_g, and $1 \in V$ since

$$\begin{aligned} V &= \{h \in G : (g,h) \in \Omega \,\text{and}\, p(g,h) \in \Omega_{\iota(g)}\} \\ &= \{h \in G : (g,h) \in \Omega \,\text{and}\, (\iota(g), p(g,h)) \in \Omega\} \end{aligned}$$

Let $W := L_g(V)$. As $W = L_g(L_g^{-1}(\Omega_{\iota(g)}))$, it is included in $\Omega_{\iota(g)}$. The set W is open because we also have $W = L_{\iota(g)}^{-1}(V)$, so we can use the continuity of $L_{\iota(g)}$.
At the end, and because $V \subseteq \Omega_g$, $W \subseteq \Omega_{\iota(g)}$, the products are defined and we can see that $L_{g|V}$ and $L_{\iota(g)|W}$ are inverses of each other. □

Definition 1.15. *A* ***sublocal group*** *of a local group $G = (G, 1, \iota, p)$ is a set $H \subseteq G$ containing 1 for which there exists an open neighborhood V of 1 in G such that*

(i) $H \subseteq V$ and H is closed in V
(ii) If $x \in H \cap \Lambda$ and $\iota(x) \in V$, then $\iota(x) \in H$
(iii) If $(x,y) \in (H \times H) \cap \Omega$ and $p(x,y) \in V$, then $p(x,y) \in H$

With H and V as above, we call H a sublocal group of G with ***associated neighborhood*** *V.*

Definition 1.16. *A* ***subgroup*** *of a local group $G = (G, 1, \iota, p)$ is a subset $H \subseteq G$ such that $1 \in H$, $H \subseteq \Lambda$, $H \times H \subseteq \Omega$ and for all $x, y \in H$, $\iota(x) \in H$ and $p(x,y) \in H$.*

Note that the subgroup H of Definition 1.16 is an actual group inside the local group G.

Definition 1.17 (**NSS, NSCS**). *A local group has no small subgroup (respectively no small connected subgroup), if it contains a neighborhood of the identity element which contains no non-trivial subgroup (resp. no non-trivial connected subgroup). We will abbreviate it* ***NSS*** *(resp.* ***NSCS****).*

Theorem 1.18. *Every Lie group has no small subgroup.*

PROOF. See for example [**18**] p.41. □

DEFINITION 1.19. *Let $G = (G, 1, \iota, p)$ be a local group.*

1. *Let U be an open neighborhood of 1 in G. Then* ***the restriction of*** *G* ***to*** *U is the local group $G|U := (U, 1, \iota|\Lambda_U, p|\Omega_U)$, where*

   $$\Lambda_U := \Lambda \cap U \cap \iota^{-1}(U) \ \ and \ \ \Omega_U := \Omega \cap (U \times U) \cap p^{-1}(U)$$

   A ***restriction of*** *G is a local group $G|U$ where U is an open neighborhood of 1 in G.*
2. *G is* ***globalizable*** *if there is a topological group H and an open neighborhood U of 1_H in H such that $G = H|U$.*

As written in [**22**], examples of local groups which are not globalizable are easy to find, because global groups, and, hence, their restrictions, always satisfy (with the convention that statements involving an undefined formula are false):

1. (cancellation law) $\forall g \forall h \forall k (gk = hk \Rightarrow g = h) \wedge (kg = kh \Rightarrow g = h)$
2. (inversion law) $\forall g \forall h (gh)^{-1} = h^{-1}g^{-1}$
3. (involution law) $\forall g (g^{-1})^{-1} = g$

so it suffices to find a local group $G = (G, 1, ^{-1}, .)$ which does not obey one of the above laws: in that case it cannot be the restriction of a global group. The following example, from [**14**], involves a local group which cannot be the restriction of a global group because it lacks a "global" inversion law.

EXAMPLE 1.20. *Let $G = \mathbb{R}$, and let the identity element be 0. The formulas for the multiplication and inversion maps are defined as follows:*

$$p(x,y) := \frac{2xy - x - y}{xy - 1} \quad \iota(x) := \frac{x}{2x - 1}$$

We use

$$\Lambda := \{x : x \neq 1/2, x \neq 1\} \subseteq \mathbb{R}$$

$$\Omega := \{(x,y) : |xy| \neq 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

as the domains of definition of ι and p respectively.

However, in this situation, the group element $1 \in \mathbb{R}$ plays a strange role. We find $p(x,1) = p(1,x) = 1$ for all $x \neq 1$. Moreover, its 'inverse' $\iota(1) = 1$ can even be defined, although the product of the two, $p(1, \iota(1))$, is not defined, and hence not equal to the identity. One can also note that this infinite group element is 'inaccessible', in the sense that it is not the product of two elements different from itself, as $p(x,y) = 1$ iff either $x = 1$ or $y = 1$. In particular, the cancellation law is not satisfied.

DEFINITION 1.21. *Let $G := (G, 1, p, \iota)$ be a local group. G is* ***a local Lie group*** *if G is a smooth manifold such that the maps ι and p are smooth.*

Back to Example 1.20, let $U := \{x : |x| < 1/2\}$. Then $G|U$ is an abelian, local Lie group, which can be globalized into the global Lie group $(\mathbb{R}, +, 0)$: The map $\phi : G \to \mathbb{R}$ given by $\phi(x) = x/(x-1)$ satisfies $\phi(p(x,y)) = \phi(x) + \phi(y)$, and $\phi(\iota(x)) = -\phi(x)$ where defined. Therefore ϕ provides the desired local group homeomorphism mapping $G|U$ to the open interval $N = \{-1 < x < 1/3\} \subseteq \mathbb{R}$.

Theorem 1.22 (**Cartan**). *Every local Lie group has a restriction which is a restriction of a global Lie group.*

3. Statement of the local H5

Now here are two forms of the local Hilbert's fifth problem (H5)

Local H5-First Form : If G is a locally euclidean local group, then some restriction of G is a local Lie group.

Local H5-Second Form : If G is a locally euclidean local group, then some restriction of G is globalizable.

As explained in [**5**], the equivalence of the two forms can be seen in the following manner : by Theorem 1.22, if G is a local Lie group, then some restriction of G is equal to some restriction of a Lie group. Thus the first form implies the second form. Conversely, by the Montgomery-Zippin-Gleason solution to the original H5, if U is open in G and such that $G|U$ is globalizable, the global group containing $G|U$ will be locally euclidean and there will be a smooth structure on it making it a Lie group. Then $G|U$ will be a local Lie group.

4. Associativity

From now on, products of two elements x, y are denoted $x.y$ or xy, and inverse x^{-1}.

Definition 1.23. *A local group G is* ***globally associative****, if, given any finite sequence of elements from G, if there are two ways of introducing parentheses such that both products thus formed exist, then the two products are in fact equal. It is* ***globally inversional*** *if the inversion map is defined everywhere, so that $\Lambda = G$.*

A theorem of Mal'cev states that **a (globally inversional) local group is globalizable iff it is globally associative**.

Jacoby, in [**10**], claims in his theorem 8 that every local group is globally associative, but in fact this is not always the case. Unfortunately the solution of local H5 was based on his theorem 8.

In paper [**14**], Olver constructs local Lie groups which are associative up to sequences of length n for a given n but which are not associative for sequences of length $n+1$.
The intuition that a local group can be viewed as an object that behaves like a group near the identity, but for which the group laws can break down once one moves far enough away from the identity, can be, and by the way global associativity, formalized as follows : Let us say that a word $g_1 \dots g_n$ in a local group G is **defined** in G if every possible way of associating this word using parentheses is well defined from applying the product operation:

Definition 1.24. *Let $a_1, \dots, a_n, b \in G$ with $n \geq 1$. We define the notion $(a_1, \dots, a_n)$ **represents** b, denoted $(a_1, \dots, a_n) \to b$, by induction on n as follows :*

(i) $(a_1) \to b$ *iff* $a_1 = b$
(ii) $(a_1, \dots, a_{n+1}) \to b$ *iff for every* $i \in \{1, \dots, n\}$*, there exists* $b_i^{'}, b_i^{''} \in G$ *such that* $(a_1, \dots, a_i) \to b_i^{'}$, $(a_{i+1}, \dots, a_{n+1}) \to b_i^{''}$, $(b_i^{'}, b_i^{''}) \in \Omega$ *and* $b_i^{'}.b_i^{''} = b$.

By convention, we say that $(a_1, \dots, a_n)$ represents 1 when $n = 0$.
We say that $a_1 \dots a_n$ is defined if there is $b \in G$ such that $(a_1, \dots, a_n)$ represents b ; in that case we write $a_1 \dots a_n$ for this (necessarily unique) b.

For instance, in order for $abcd$ to be defined, $((ab)c)d$, $(a(bc))d$,$(ab)(cd)$, $a(b(cd))$ and $a((bc)d)$ must all be well defined.
As an example, within $(\{-9, \dots, 9\}, 0, +, -)$, we have $(-2+6)+5$ which is defined, but $-2+(6+5)$ which is not defined.

Back to Example 1.20, we see that the products $\frac{1}{2} \cdot \frac{1}{2}$, $\frac{1}{2} \cdot 3 = 3 \cdot \frac{1}{2}$ and $\frac{1}{2} \cdot (\frac{1}{2} \cdot 3)$, $(\frac{1}{2} \cdot \frac{1}{2}) \cdot 3 = 3 \cdot (\frac{1}{2} \cdot \frac{1}{2})$ are all defined, and satisfy the local associative law (2.2):

$$\frac{1}{2} \cdot (\frac{1}{2} \cdot 3) = \frac{1}{3} = (\frac{1}{2} \cdot \frac{1}{2}) \cdot 3$$

However, as $\frac{1}{2} \cdot 3 = -1$, neither the products $3((\frac{1}{2} \cdot \frac{1}{2})3)$, $(3 \cdot \frac{1}{2}) \cdot (\frac{1}{2} \cdot 3)$, nor $3(\frac{1}{2}(\frac{1}{2} \cdot 3))$, etc... are defined.

Further with Example 1.20, one can consider a product involving $2, 3, 4, 5$. The products $((2.3)4)5$, $(2(3.4))5$, $(2.3)(4.5)$, $2(3(4.5))$ and $2((3.4)5)$ are all defined, but $(2.3)(4.5) = \frac{89}{77}$, while $((2.3)4)5 = \frac{73}{61}$, hence $2.3.4.5$ is not defined.

The concept of relative closeness is thus useful.

Lemma 1.25. *Let G be a local group. There are open symmetric neighborhoods U_n of 1 for $n > 0$ such that $U_{n+1} \subseteq U_n$ and for all $(a_1, \dots, a_n) \in U_n^{\times n}$, $a_1 \dots a_n$ is defined.*

PROOF. Let $U \subseteq \Lambda$ be an open neighborhood of 1 in G. Then $U \cap U^{-1} =: U_1$ is a symmetric open neighborhood of 1. Since by definition of a local group, the map $p : \Omega \to G$ is continuous, and since $1 \in image(p) \cap U_1$, there are open sets V_1, V_2 such that $V_1 \times V_2 \subseteq \Omega$ and $V_1 V_2 \subseteq U_1$. We let $U_2 := V_1 \cap V_1^{-1} \cap V_2 \cap V_2^{-1}$, then U_2 is symmetric and $U_2 \subseteq V_1 \cap V_2$ so $U_2 \times U_2 \subseteq \Omega$ and $U_2 U_2 \subseteq V_1 V_2 \subseteq U_1$.

Now assume inductively that $n \geq 2$ and that for $m = 1, \dots, n$,

(i) U_m is a symmetric open neighborhood of 1,
(ii) $U_{m+1} \subseteq U_m$ if $m < n$,
(iii) for all $(a_1, \dots, a_m) \in U_m^{\times m}$, $a_1 \dots a_m$ is defined,
(iv) the map $\phi_m : U_m^{\times m} \to G$ defined by $\phi_m(a_1, \dots, a_m) = a_1 \dots a_m$ is continuous.

Let U_{n+1} be a symmetric open neighborhood of 1 such that $U_{n+1} \subseteq U_n$ and such that, by continuity of ϕ_n, $U_{n+1}^{\times n} \subseteq \phi_n^{-1}(U_2)$. Now let $(a_1, \dots, a_{n+1}) \in U_{n+1}^{\times(n+1)}$, we want to show that $(a_1, \dots, a_{n+1})$ represents $a_1.(a_2, \dots, a_{n+1})$. As $a_1 \in U_{n+1} \subseteq U_2$ and $a_2 \dots a_{n+1}$ is defined by induction hypothesis and belongs to U_2, as shown above, we get $(a_1, (a_2 \dots a_{n+1})) \in \Omega$. If $3 \leq k \leq n$,

$$a_1.((a_2 \dots a_k).(a_{k+1} \dots a_{n+1}))$$

is defined and equals

$$(a_1.(a_2 \dots a_k)).(a_{k+1} \dots a_{n+1}))$$

because all the couples involved in products belong to Ω and so we can use Formula (2.2) from Definition 1.12. Note also that ϕ_{n+1} is continuous. $\square$

The beginning of the proof also show a local version of Proposition 1.7, namely that if G is a local group, $\Lambda = G$ and V an open neighborhood of 1 in G, then there exists a symmetric open neighborhood W of 1 such that $W.W \subseteq V$.

COROLLARY 1.26. *The map* $\phi_n : U_n^{\times n} \to G$ *defined by* $\phi_n(a_1, \dots, a_n) = a_1 \dots a_n$ *is continuous, and* $U_{n+1}^{\times n} \subseteq \phi_n^{-1}(U_2)$.

If $A \subseteq U_n$, let $A^n := \{a_1 \dots a_n : (a_1, \dots, a_n) \in A^{\times n}\}$.

Recall that if U is an open neighborhood of 1, we have set $\Omega_U := \Omega \cap (U \times U) \cap p^{-1}(U)$. The construction of the sets U_n now allow us to state the following result.

COROLLARY 1.27. *Let* $G = (G, 1, ^{-1}, .)$ *be a local group. Suppose* $\Lambda = G$. *Let* $g, h \in U_3$ *such that* $(g, h) \in \Omega_{U_3}$. *Then* $(h^{-1}, g^{-1}) \in \Omega_{U_3}$ *and* $(gh)^{-1} = h^{-1}g^{-1}$.

PROOF. Suppose $g, h \in U_3$ with $(g, h) \in \Omega_{U_3}$. We thus have

$$h^{-1}, g^{-1}, gh \in U_3$$

so $h^{-1}.g^{-1}.(gh)$ is defined and

$$h^{-1}.g^{-1}.(gh) = h^{-1}.(g^{-1}.(gh)) = h^{-1}h = 1$$

because of (2.2). We have also

$$h^{-1}.g^{-1}.(gh) = (h^{-1}g^{-1}).(gh)$$

so $h^{-1}g^{-1} = (gh)^{-1}$. The latter is contained in U_3, thus $(h^{-1}, g^{-1}) \in \Omega_{U_3}$. □

We next look at some assumptions that it is possible to make on the local group G without loss of generality. Set $U := \Lambda \cap \Lambda^{-1}$. Then U is an open neighborhood of 1 in G. If $g \in U$, then $g^{-1} \in \Lambda$ and $(g^{-1})^{-1} = g \in \Lambda$. Thus $\Lambda \cap U \cap U^{-1} = U$, meaning that the open set Λ_U defined in 1.19 is equal to U.

Hence, **from now on, we suppose** $\Lambda = G$, as every local group has a restriction satisfying this condition.

In particular G is now symmetric and we have the property that if $(x, y) \in \Omega$ and $xy = 1$, then $x = y^{-1}$ and $y = x^{-1}$. Thus, for all $x \in G$, $(x^{-1})^{-1} = x$.

Furthermore, Corollary 1.27 tells us that every local group has a restriction satisfying the following assumption :

$$\text{if } (g, h) \in \Omega \text{ then } (h^{-1}, g^{-1}) \in \Omega \text{ and } (gh)^{-1} = h^{-1}g^{-1}$$

which will be made here.

In other words, the involution law and the inversion law are satisfied.

We will use the next lemma repeatedly. Its proof is not difficult but requires some care.

LEMMA 1.28. *Let $a, a_1, \ldots, a_n \in G$. Then*

1. *If $a_1 \ldots a_n$ is defined and $1 \le i \le j \le n$, then $a_i \ldots a_j$ is defined. In particular, if a^n is defined and $m \le n$, then a^m is defined.*
2. *If a^m is defined and $i, j \in \{0, \ldots, m\}$ are such that $i + j = m$, then $(a^i, a^j) \in \Omega$ and $a^i.a^j = a^m$.*
3. *If a^n is defined and, for all $i, j \in \{1, \ldots, n\}$ with $i + j = n + 1$, one has $(a^i, a^j) \in \Omega$, then a^{n+1} is defined.*
   More generally, if $a_1 \ldots a_n$ is defined, $a_i \ldots a_{n+1}$ is defined for all $i \in \{2, \ldots, n\}$ and
   $$(a_1 \ldots a_i, a_{i+1} \ldots a_{n+1}) \in \Omega \text{ for all } i \in \{1, \ldots, n\}$$
   then $a_1 \ldots a_{n+1}$ is defined.
4. *If a^n is defined, then $(a^{-1})^n$ is defined and $(a^{-1})^n = (a^n)^{-1}$. We denote $(a^{-1})^n$ by a^{-n}.*

More generally, if $a_1 \dots a_n$ *is defined, then* $a_n^{-1} \dots a_1^{-1}$ *is defined and* $(a_1 \dots a_n)^{-1} = a_n^{-1} \dots a_1^{-1}$.

(5) If $k, l \in \mathbb{Z}$, $l \neq 0$, *and* $a^{k.l}$ *is defined, then* a^k *is defined,* $(a^k)^l$ *is defined and* $(a^k)^l = a^{k.l}$.

(6) If $i, j \in \mathbb{Z}$ *and* $i.j < 0$, *if* a^i *and* a^j *are defined and* $(a^i, a^j) \in \Omega$, *then* a^{i+j} *is defined and* $a^i.a^j = a^{i+j}$.

Proof. (1) Immediate from the definitions

(2) Immediate from the definitions

(3) Follows from repeated use of Formula (2.2) (from Definition 1.12, item local associativity)

(4) We prove the first assertion by induction on n. We have supposed that $\Lambda = G$, so the case $n = 1$ is trivial. The case $n = 2$ follows from Corollary 1.27 : if $(a, a) \in \Omega$, then $(a^{-1}, a^{-1}) \in \Omega$ and $(a^2)^{-1} = a^{-1}.a^{-1} = (a^{-1})^2$.
For the induction step, suppose a^{n+1} is defined and $i, j \in \{1, \dots, n\}$ with $i + j = n + 1$. As $(a^j, a^i) \in \Omega$, we get by induction

$$((a^{-1})^i, (a^{-1})^j) \in \Omega$$

Thus, by (3) of the lemma, we know that $(a^{-1})^{n+1}$ is defined, and, using the induction hypothesis, that :

$$(a^{-1})^{n+1} = (a^{-1})^n.a^{-1} = (a^n)^{-1}.a^{-1} = (a.a^n)^{-1} = (a^{n+1})^{-1}$$

The second part of (4) is shown similarly :
The cases $n = 1$ and $n = 2$ are similar to those of the first assertion. For the induction step, suppose $a_1 \dots a_{n+1}$ is defined. Then, for $i, j \in \{1, \dots, n\}$ such that $i + j = n + 1$,

$$(a_1 \dots a_i, a_{i+1} \dots a_{n+1}) \in \Omega$$

hence

$$((a_{i+1} \dots a_{n+1})^{-1}, (a_1 \dots a_i)^{-1}) \in \Omega$$

and

$$(a_1 \dots a_{n+1})^{-1} = a_{n+1}^{-1} \dots a_1^{-1}$$

(5) It is enough to check the assertion for $k, l \in \mathbb{N}$, $l \neq 0$, since we can then use (4) of the lemma : a^k being defined for $k < 0$ means that a^{-k} is defined. By (1) of the lemma and because $k \leq k.l$, we have that a^k is defined. Then, we use induction on l. Suppose $l \geq 2$, the assertion is true for $i \leq l$ and $a^{k.(l+1)}$ is defined.
In order to see that $(a^k)^{l+1}$ is defined, one must check that

$$((a^k)^i, (a^k)^j) \in \Omega$$

for $i, j \in \{1, \dots, l\}$ such that $i + j = l + 1$.
This is the case by (2), because for such i, j, we have $k.i + k.j =$

$k.(l+1)$ and by induction $(a^k)^i = a^{k.i}$ and $(a^k)^j = a^{k.j}$. So $(a^k)^{l+1}$ is defined, and, by induction, we get :

$$(a^k)^{l+1} = (a^k)^l.a^k = a^{k.l}.a^k = a^{k.l+k}$$

(6) From the previous assertions we obtain that a^{i+j} and a^{-j} are defined. Then, if $i = -j$, the result is obvious. Let i, j be such that $i > 0$, $j < 0$ and $i > |j|$. By the other assertions we have $a^i = a^{i+j}.a^{-j}$, whence $a^i.a^j = a^{i+j}$ by (2.2):

$$a^i.a^j = (a^{i+j}.a^{-j})a^j = a^{i+j}.(a^{-j}.a^j) = a^{i+j}$$

The other cases are similar.

□

5. Group germs

Studying local groups, it seems natural to consider restrictions of them, and hence classes of restrictions of them:

Definition 1.29. *Let $G = (G, 1, \iota, p)$ and $G' = (G', 1', \iota', p')$ be two local groups with domains respectively Ω, Λ and Ω', Λ'. They are called locally identical if they have a common restriction, i.e if there exists a set $U \subseteq G \cap G'$ such that $G|U = G'|U$ ($1 = 1'$, and the topology and the group operations as well as their domains agree on U) .*

It is obviously an equivalence relation; an equivalence class $[G]$ of local groups is called a germ or a group germ.

In the next two sections, we will see how to construct an ultrapower of a local group G, in which we will consider a set called the monad of the identity element 1. This set is closely related to the group germ $[G]$, but has the advantage to be a (global) group, instead of an equivalence class of local groups.

CHAPTER 2

Saturation and ultraproducts

In this section we see how to construct a saturated ultraproduct which is an elementary extension of a structure suiting our case. We first recall the model-theoretic framework, then define the notions of type and of saturation, a property allowing to go from a statement of the form $\forall x \exists y \phi$ to a statement of the form $\exists y \forall x \psi$; this kind of arrangement of quantifiers being particularly useful when dealing with limits and asymptotic behaviours. We then define ultraproducts, and ultrapowers, which are particular cases of elementary extensions of a given model. We then look at necessary conditions of construction. The section is mainly based on [**3**], and on [**12**].

1. Model theory : basic definitions

Definition 2.1. *A **language** $\mathcal{L}$ is given by :*

(1) A set of function symbols $\mathcal{F}$, and positive integers n_f (arities) for each $f \in \mathcal{F}$.
(2) A set of relation symbols $\mathcal{R}$, and positive integers n_R (arities) for each $R \in \mathcal{R}$.
(3) A set of constant symbols $\mathcal{C}$

We write $\mathcal{L} = \{f_1, f_2, \ldots, R_1, R_2, \ldots, c_1, c_2, \ldots\}$

Definition 2.2. *An **$\mathcal{L}$-structure** $\mathcal{M}$ is given by :*

(1) A nonempty set M, called the universe, domain, or underlying set of $\mathcal{M}$.
(2) A function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$.
(3) A set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$.
(4) An element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.

We write $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$.

Definition 2.3. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures with domains M, N. An **$\mathcal{L}$-embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that preserves the interpretation of all the symbols of $\mathcal{L}$.*

(1) For all $f \in \mathcal{F}$ and $a_1, \ldots, a_{n_f} \in M$,

$$\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}))$$

(2) For all $R \in \mathcal{R}$ and $a_1, \ldots, a_{m_R} \in M$,

$$(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}} \Leftrightarrow (\eta(a_1), \ldots, \eta(a_{m_R})) \in R^{\mathcal{N}}$$

(3) For all $c \in \mathcal{C}$, $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-isomorphism. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either that $\mathcal{M}$ is a substructure of $\mathcal{N}$ or that $\mathcal{N}$ is an extension of $\mathcal{M}$.

DEFINITION 2.4. *The set of **$\mathcal{L}$-terms** is the smallest set τ such that:*

(1) $c \in \tau$ for each constant symbol $c \in \mathcal{C}$.
(2) τ contains variable symbols v_i for $i = 1, 2, \ldots$.
(3) If $t_1, \ldots, t_{n_f} \in \tau$, then $f(t_1, \ldots, t_{n_f}) \in \tau$ for each $f \in \mathcal{F}$.

DEFINITION 2.5. *An atomic **$\mathcal{L}$-formula** ϕ is either:*

(1) $t_1 = t_2$, where $t_1, t_2 \in \tau$
(2) $(t_1, \ldots, t_{n_R}) \in R$, where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R} \in \tau$. We usually write it $R(t_1, \ldots, t_{n_R})$.

The set of $\mathcal{L}$-formulas is the smallest set $\mathcal{W}$ containing the atomic formulas and such that :

(1) If $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$.
(2) If ϕ and $\psi \in \mathcal{W}$, then $(\phi \wedge \psi) \in \mathcal{W}$ and $(\phi \vee \psi) \in \mathcal{W}$.
(3) If $\phi \in \mathcal{W}$, then $\exists v_i \phi \in \mathcal{W}$ and $\forall v_i \phi \in \mathcal{W}$.

*We say that a variable v occurs freely in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier, otherwise we say that it is bound. A **sentence** is a formula without free variable.*

DEFINITION 2.6 (**satisfiability**). *Let ϕ be a formula with free variables from $\overline{v} = (v_{i_1}, \ldots, v_{i_m})$. Let $\overline{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \vDash \phi(\overline{a})$ as follows :*

(1) If ϕ is $t_1 = t_2$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$
(2) If ϕ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_{n_R}^{\mathcal{M}}(\overline{a})) \in R^{\mathcal{M}}$
(3) If ϕ is $\neg\psi$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $\mathcal{M} \nvDash \psi(\overline{a})$
(4) If ϕ is $(\psi \vee \theta)$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $\mathcal{M} \vDash \psi(\overline{a})$ or $\mathcal{M} \vDash \theta(\overline{a})$
(5) If ϕ is $(\psi \wedge \theta)$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $\mathcal{M} \vDash \psi(\overline{a})$ and $\mathcal{M} \vDash \theta(\overline{a})$
(6) If ϕ is $\exists v_j \psi(\overline{v}, v_j)$, then $\mathcal{M} \vDash \phi(\overline{a})$ if there is $b \in M$ such that $\mathcal{M} \vDash \psi(\overline{a}, b)$
(7) If ϕ is $\forall v_j \psi(\overline{v}, v_j)$, then $\mathcal{M} \vDash \phi(\overline{a})$ if $\mathcal{M} \vDash \psi(\overline{a}, b)$ for all $b \in M$

If $\mathcal{M} \vDash \phi(\overline{a})$ we say that $\mathcal{M}$ satisfies $\phi(\overline{a})$ or that $\phi(\overline{a})$ is true in $\mathcal{M}$.

An $\mathcal{L}$-theory T is simply a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a model of T, and write $\mathcal{M} \vDash T$, if $\mathcal{M} \vDash \phi$ for all sentences $\phi \in T$. A theory is said to be satisfiable if it has a model.

If T is an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, we say that ϕ is a logical consequence of T and write $T \vDash \phi$, if whenever $\mathcal{M} \vDash T$, then $\mathcal{M} \vDash \phi$. An $\mathcal{L}$-theory T is called complete if for any $\mathcal{L}$-sentence ϕ, either $T \vDash \phi$ or $T \vDash \neg\phi$.

$Th(\mathcal{M})$ denotes the full theory of $\mathcal{M}$, i.e. the set of $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \vDash \phi$. Note that $Th(\mathcal{M})$ is a complete theory.

A set Σ of $\mathcal{L}$-sentences is said to be **finitely satisfiable** iff every finite subset of Σ is satisfiable.

THEOREM 2.7 (**Compactness Theorem**). *A set Σ of $\mathcal{L}$-sentences is finitely satisfiable iff it is satisfiable.*

See for example [**3**] for a proof.

DEFINITION 2.8 (**Definable sets**). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and $A \subseteq M$. We say that $X \subseteq M^n$ is A-definable iff there is $\overline{b} \in A^m$ and an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ such that:*

$$X = \{\overline{a} \in M^n : \mathcal{M} \vDash \phi(\overline{a}, \overline{b})\}$$

We say that $\phi(\overline{v}, \overline{b})$ defines X, and that X is definable without parameters if it is $\emptyset$-definable.

DEFINITION 2.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called an **elementary embedding** if, whenever $a_1, \ldots, a_n \in M$, and $\phi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, then*

$$\mathcal{M} \vDash \phi(a_1, \ldots, a_n) \Leftrightarrow \mathcal{N} \vDash \phi(j(a_1), \ldots, j(a_n))$$

If $\mathcal{M}$ is a substructure of $\mathcal{N}$ we then say that it is an elementary substructure, or that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, and write $\mathcal{M} \prec \mathcal{N}$.

2. Types and saturation

Let $\mathcal{L}$ be a language, T a satisfiable $\mathcal{L}$-theory, and let $\mathcal{L}_{\overline{x}} := \mathcal{L} \cup \{x_1, \ldots, x_n\}$; $x_1, \ldots, x_n$ being new constant symbols, denoted like variables by commodity. Let $S_n(T)$ be the set of complete $\mathcal{L}_{\overline{x}}$-theories containing T. An element of $S_n(T)$ is called a **complete** n**-type**. A set of $\mathcal{L}$-formulas with n free variables is a n**-type** if it can be completed in a complete n-type.

Let $\mathcal{M} \vDash T$ and $A \subseteq M$. Adding to $\mathcal{L}$ one new constant symbol for each element of A, we obtain the language $\mathcal{L}_A$. We let $Th_A(\mathcal{M})$ denote the theory of $\mathcal{M}$ in the language $\mathcal{L}_A$; and $S_n^{\mathcal{M}}(A) := S_n(Th_A(\mathcal{M}))$.

Let $\overline{c} \in M^n$. The set of $\mathcal{L}_A$-formulas $\phi(v_1, \ldots, v_n)$ for which $\mathcal{M} \vDash \phi(\overline{c})$ is denoted $tp_A^{\mathcal{M}}(\overline{c})$ or $p_A^{\mathcal{M}}(\overline{c})$. For every type $p(\overline{x}) \in S_n(T)$, there exists $\mathcal{M} \vDash T$ and $\overline{a} \in M^n$ such that $tp^{\mathcal{M}}(\overline{a})$ is equivalent to $p(\overline{x})$.

DEFINITION 2.10 (**κ-saturated, saturated**). *Let κ be an infinite cardinal, $\mathcal{L}$ a language such that $|\mathcal{L}| < \kappa$, T a satisfiable $\mathcal{L}$-theory, and $\mathcal{M} \vDash T$. We say that $\mathcal{M}$ is κ-saturated if, whenever $A \subseteq M$, $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$.*
We say that $\mathcal{M}$ is saturated if it is $|M|$-saturated.

3. Ultrafilters, ultraproducts and the Transfer Principle

DEFINITION 2.11. *Let I be a nonempty set, and let $\mathcal{P}(I)$ denote the set of all subsets of I. A **filter** D over I is defined to be a set $D \subseteq \mathcal{P}(I)$ such that :*

- $I \in D$
- *If $X, Y \in D$, then $X \cap Y \in D$*
- *If $X \in D$ and $X \subseteq Z \subseteq I$, then $Z \in D$.*

The first condition is sometimes replaced by $\emptyset \notin D$ in the literature. Note that the filter D is nonempty since $I \in D$. The filter D is called trivial when it equals $\{I\}$; it is said to be proper when it is different from $\mathcal{P}(I)$.

Let $E \subseteq \mathcal{P}(I)$. The **filter generated by** E is the intersection of all filters over I including E.

A filter D over a set I is said to be **uniform** iff every member of D has the same cardinality $|I|$.

E has **the finite intersection property** iff the intersection of any finite number of elements of E is nonempty.

A filter D over I is said to be an **ultrafilter** iff for all $X \in \mathcal{P}(I)$,

$$X \in D \Leftrightarrow I \setminus X \notin D$$

Equivalently, D is an ultrafilter iff it is a **maximal proper filter**. Note that an ultrafilter can be thought of as a finitely additive two-valued measure on the subsets of I.

PROPOSITION 2.12 (**Ultrafilter Theorem**). *If $E \subseteq \mathcal{P}(I)$ and E has the finite intersection property, then there exists an ultrafilter D over I such that $E \subseteq D$. (Or, equivalently, any proper filter over I can be extended to an ultrafilter over I.*

Let $\mathcal{L}$ be a language (whose set of constants is $\mathcal{C}$, set of relations $\mathcal{R}$, set of functions $\mathcal{F}$), I an infinite set, D an ultrafilter over I, and, for each $i \in I$, let $\mathcal{M}_i$ be an $\mathcal{L}$-structure.
We define a new structure $\mathcal{M} := \prod \mathcal{M}_i/D$, called the ultraproduct of the $\mathcal{M}_i$ using D. It will also be denoted $\mathcal{M} := \prod_D \mathcal{M}_i$. When all the $\mathcal{M}_i$'s are the same structure $\mathcal{M}_0$, the ultraproduct is called an ultrapower and is often denoted $\mathcal{M}_0^*$.

First, let X be the cartesian product of the domains :

$$X := \prod_{i \in I} M_i = \{f : I \to \bigcup_{i \in I} M_i : \forall i, f(i) \in M_i\}$$

We then define a relation $\approx$ on X :

$$f \approx g \text{ iff } \{i \in I : f(i) = g(i)\} \in D$$

It is easy to see that $\approx$ is an equivalence relation.
We define the universe of $\mathcal{M}$ to be $M := X/\approx$.

$\mathcal{M}$ is an $\mathcal{L}$-structure :

- If $c \in \mathcal{C}$, let $c^{\mathcal{M}}$ be the equivalence class of $f_c \in X$, where $f_c(i) = c^{\mathcal{M}_i}$ for all $i \in I$
- If $f \in \mathcal{F}$, with arity n, let $g_1, \ldots, g_n, h_1, \ldots, h_n \in X$ such that for $k = 1, \ldots, n$, $g_k \approx h_k$. For $i \in I$, let

  $$g_{n+1}(i) := f^{\mathcal{M}_i}(g_1(i), \ldots, g_n(i)) \text{ and}$$

  $$h_{n+1}(i) := f^{\mathcal{M}_i}(h_1(i), \ldots, h_n(i))$$

  As $g_{n+1} \approx h_{n+1}$,

  $$f^{\mathcal{M}}(g_1/\approx, \ldots, g_n/\approx) := g_{n+1}/\approx$$

  is a well defined function on $\mathcal{M}$.
- Let $R \in \mathcal{R}$ with arity n, and $g_1, \ldots, g_n, h_1, \ldots, h_n$ as above. As

  $$\{i \in I : (g_1(i), \ldots, g_n(i)) \in R^{\mathcal{M}_i}\} \in D$$

  if and only if

  $$\{i \in I : (h_1(i), \ldots, h_n(i)) \in R^{\mathcal{M}_i}\} \in D$$

  we can define

$$R^{\mathcal{M}} := \{(g_1/\approx, \ldots, g_n/\approx) : \{i \in I : (g_1(i), \ldots, g_n(i)) \in R^{\mathcal{M}_i}\} \in D\}$$

The following theorem is known as **Łòs' theorem**, but also under the denominations : **Transfer Principle**, or **Fundamental Theorem of Ultraproducts**:

THEOREM 2.13. *Suppose that I is an infinite set and D is an ultrafilter over I. Let $(\mathcal{M}_i)_{i \in I}$ be $\mathcal{L}$-structures, $\mathcal{M} := \prod \mathcal{M}_i/D$, and $\phi(v_1, \ldots, v_n)$ be a $\mathcal{L}$-formula. Then,*

$$\mathcal{M} \vDash \phi(g_1/\approx, \ldots, g_n/\approx) \textit{ iff } \{i \in I : \mathcal{M}_i \vDash \phi(g_1(i), \ldots, g_n(i))\} \in D$$

PROOF. See for example [**3**]. □

COROLLARY 2.14. *Let $\mathcal{M}^*$ be an ultrapower of $\mathcal{M}$. Then $\mathcal{M}$ is an elementary substructure of $\mathcal{M}^*$.*

4. Construction of a suitable saturated ultraproduct

We want to show the existence of κ-saturated ultraproducts for cardinals κ bigger than $\aleph_1$. In this subsection we define properties of ultrafilters such as goodness and κ -regularity. Then we show the existence of an ultrafilter with both properties, and, finally, we show that an ultraproduct modulo such an ultrafilter is κ-saturated. The reader who is mainly interested in local H5 can skip this section. This section is based on *Model Theory*, by C.C.Chang and H.G.Keisler ([**3**]). We give here more detailed proofs than in the book; except for a couple of them.

4.1. κ-regular ultrafilters.

DEFINITION 2.15. *An ultrafilter D is said to be* ***countably incomplete*** *iff D is not closed under countable intersections.*
*Let κ be a cardinal. A proper filter D over a set I is said to be κ-**regular** iff there exists a set $E \subseteq D$ of cardinality κ such that each $i \in I$ belongs to only finitely many $e \in E$.*

Note that an ultrafilter D is ω-regular iff D is countably incomplete.

THEOREM 2.16. *Let $\mathcal{L}$ be countable, and let D be a countably incomplete ultrafilter over a set I. Then for every family $\mathcal{M}_i$, $i \in I$, of models for $\mathcal{L}$, the ultraproduct $\prod_D \mathcal{M}_i$ is $\aleph_1$-saturated.*

PROPOSITION 2.17 (Existence). *For any set I of infinite cardinality κ, there exists a κ-regular ultrafilter D over I.*

PROOF. [**3**]It suffices to show that some set J of cardinality κ has a κ-regular ultrafilter over it ; hence we can consider the set of all finite subsets of κ, namely $\mathcal{P}_\omega(\kappa)$ (as it has cardinality κ). For each $\beta \in \kappa$, let $\hat{\beta} := \{j \in \mathcal{P}_\omega(\kappa) : \beta \in j\}$, and let $E := \{\hat{\beta} : \beta \in \kappa\}$. Then $|E| = \kappa$. Moreover, each $j \in \mathcal{P}_\omega(\kappa)$ belongs to only finitely many $\hat{\beta} \in E$, because j is finite, and $j \in \hat{\beta}$ means $\beta \in j$. It follows that any proper filter over $\mathcal{P}_\omega(\kappa)$ which includes E is κ-regular.

Let $\hat{\beta}_1, \dots, \hat{\beta}_n, n \in \omega$ be elements of E. Note that $\hat{\beta}_i$ is the set of all finite subsets of κ that contain β_i, so $\hat{\beta}_i$ contains $\{\beta_1, \dots, \beta_n\}$. Thus

$$\{\beta_1, \dots, \beta_n\} \in \hat{\beta}_1 \cap \ldots \cap \hat{\beta}_n$$

i.e. E has the finite intersection property.

Hence by the Ultrafilter Theorem 2.12, E can be extended to an ultrafilter D over $\mathcal{P}_\omega(\kappa)$, whence D is a κ-regular ultrafilter. □

4.2. Goodness.

DEFINITION 2.18. *Let I be a nonempty set and β a cardinal. We consider functions f, g on the set $\mathcal{P}_\omega(\beta)$ of all finite subsets of β into the set $\mathcal{P}(I)$ of all subsets of I.*

(1) We say that $f \leq g$ iff for all $u \in \mathcal{P}_\omega(\beta)$, $f(u) \subseteq g(u)$ (i.e. each value of f is included in the corresponding value of g).

*(2) We say that f is **monotonic** iff for $u, v \in \mathcal{P}_\omega(\beta)$,*

$$u \subseteq v \Rightarrow f(u) \supseteq f(v)$$

*(3) We say that f is **additive** iff for $u, v \in \mathcal{P}_\omega(\beta)$,*

$$f(u \cup v) = f(u) \cap f(v)$$

LEMMA 2.19. *Every additive function on $\mathcal{P}_\omega(\beta)$ into $\mathcal{P}(I)$ is monotonic.*

The proof is clear.

DEFINITION 2.20. *Let κ be an infinite cardinal. An ultrafilter D over a set I is said to be κ**-good** iff :*
For every cardinal $\beta < \kappa$ and every monotonic function $f : \mathcal{P}_\omega(\beta) \to D$, there exists an additive function $g : \mathcal{P}_\omega(\beta) \to D$ such that $g \leq f$. (monotonic functions can be refined by additive functions)

Note that if D is κ-good then D is β-good for all infinite cardinals $\beta < \kappa$.

LEMMA 2.21. *An ultrafilter D is κ^+-good iff :*
For every monotonic function $f : \mathcal{P}_\omega(\kappa) \to D$, there exists an additive function $g : \mathcal{P}_\omega(\kappa) \to D$ such that $g \leq f$.

PROOF. [**3**] The necessity is clear. Conversely, let $\beta \leq \kappa$, and let $f : \mathcal{P}_\omega(\beta) \to D$ be monotonic. We define another function F :

$$\begin{array}{rccc} F: & \mathcal{P}_\omega(\kappa) & \to & D \\ & u & \mapsto & F(u) = f(u \cap \beta) \end{array}$$

F is monotonic, so there exists an additive function $G : \mathcal{P}_\omega(\kappa) \to D$ such that $G \leq F$. Now we put

$$g := G_{\restriction \mathcal{P}_\omega(\beta)}$$

The function g maps $\mathcal{P}_\omega(\beta)$ into D, is additive, and satisfies $g \leq f$: for $u \in \mathcal{P}_\omega(\beta)$, $g(u) = G(u) \leq F(u) = f(u \cap \beta) = f(u)$. □

LEMMA 2.22. *Let X be a set of infinite cardinality κ, and let $(Y_x)_{x \in X}$ be a family of sets each of which has cardinality κ. Then there exists a family of sets $(Z_x)_{x \in X}$, such that for all $x, y \in X$:*

(1) $Z_x \subseteq Y_x$
(2) $|Z_x| = \kappa$
(3) $x \neq y \Rightarrow Z_x \cap Z_y = \emptyset$

In other words, any family of κ sets, each (of which) of cardinality κ, can be refined to a family of κ disjoint sets of cardinality κ.

PROOF. [**3**] We may assume without loss of generality that $X = \kappa$. For each ordinal $\alpha < \kappa$, we define X_α to be the set of ordered couples :

$$X_\alpha := \{(\gamma, \delta) : \gamma \leq \delta \,\text{and}\, \delta < \alpha\}$$

It is a subset of $\kappa \times \kappa$, that can be viewed as a right triangle with sides of length α. We also let

$$X_\kappa := \bigcup_{\beta<\kappa} X_\beta$$

Then we want to construct a function

$$\begin{array}{cccc} f: & X_\kappa & \to & \bigcup\{Y_\eta : \eta < \kappa\} \\ & (\gamma,\delta) & \mapsto & f(\gamma,\delta) \text{ such that } f(\gamma,\delta) \in Y_\gamma \end{array}$$

and such that f is one-one. Once this function f found, it is possible to define

$$Z_\gamma := \{f(\gamma,\delta) : \gamma \leq \delta < \kappa\}$$

and the family $(Z_\gamma)_{\gamma<\kappa}$ clearly has the three desired properties.

So we shall now define the function f, using transfinite induction : Let $\alpha < \kappa$ and suppose that we already have a one-one function f_α with domain X_α and satisfying the following property :

$$P(\alpha): \text{ whenever } \gamma \leq \delta < \alpha, f(\gamma,\delta) \in Y_\gamma \tag{4.1}$$

As $|X_\alpha| < \kappa$, and as $|Y_\gamma| = \kappa$ for all $\gamma < \kappa$, we may extend f_α to a one-one function $f_{\alpha+1}$ with domain $X_{\alpha+1}$ such that (4.1) holds for $\alpha+1$, i.e. such that $P(\alpha+1)$.
For each $\gamma \leq \alpha$, we pick a value $f_{\alpha+1}(\gamma,\alpha) \in Y_\gamma$ which is different from all the previously chosen values of $f_{\alpha+1}$ (which is possible since $|Y_\gamma| > |X_{\alpha+1}|$). By taking unions at the limit ordinals, we obtain a chain of one-one functions f_α with domain X_α and satisfying property (4.1). Then the union $f = \bigcup_{\alpha<\kappa} f_\alpha$ is a one-one function with domain X_κ satisfying (4.1) $P(\kappa)$. □

Definition 2.23. *Let Π be a nonempty collection of partitions of κ such that each partition has exactly κ equivalence classes, and let F be a nontrivial filter over κ.*

*(1) We say that the pair (Π, F) is **consistent** iff given any $X \in F$ and any $X_1, \ldots, X_n, n \in \omega$, each X_i belonging to a distinct partition $P_i \in \Pi$, then*

$$X \cap \bigcap_{1\leq i\leq n} X_i \neq \emptyset$$

(2) If F is a filter and $F \cup E$ has the finite intersection property, we let (F, E) denote the filter generated by $F \cup E$.

Lemma 2.24. *Let κ be an infinite cardinal.*

(i) Let F be a uniform filter over κ, generated by a subset $E \subseteq F$, such that $|E| \leq \kappa$.
There exists a collection Π of partitions of κ such that $|\Pi| = 2^\kappa$ and (Π, F) is consistent.

(ii) Suppose that (Π, F) is consistent. Let $J \subseteq \kappa$.
Then either $(\Pi, (F, \{J\}))$ is consistent, or else $(\Pi', (F, \{\kappa \setminus J\}))$ is consistent for some cofinite $\Pi' \subseteq \Pi$.

(iii) Suppose that (Π, F) *is consistent. Let* p *be any monotonic mapping of* $\mathcal{P}_\omega(\kappa)$ *into* F *and let* $P \in \Pi$.
Then there exists an extension F' *of* F *and an additive function* $q : \mathcal{P}_\omega(\kappa) \to F'$ *such that* $q \leq p$ *and* $(\Pi \setminus \{P\}, F')$ *is consistent.*

PROOF. (i) Let $(J_\beta)_{\beta<\kappa}$ be a list of all finite intersections of members of $E \subseteq F$. As a filter, F is closed under finite intersections, so each J_β is a member of F, and thus has cardinality κ by uniformity of F. By Lemma 2.22, there is a family $(I_\beta)_{\beta<\kappa}$, such that $|I_\beta| = \kappa$, $I_\beta \subseteq J_\beta$, and $I_\beta \cap I_{\beta'} = \emptyset$ if $\beta \neq \beta'$. Consider the set of couples :

$$B = \{(s, r) : s \in \mathcal{P}_\omega(\kappa) \,\text{and}\, r : \mathcal{P}(s) \to \kappa\}$$

Given $s \in \mathcal{P}_\omega(\kappa)$, let ${}^{\mathcal{P}(s)}\kappa$ denote the set of functions from $\mathcal{P}(s)$ to κ. Then $|B| = |\mathcal{P}_\omega(\kappa)| \times |{}^{\mathcal{P}(s)}\kappa| = \kappa \times \kappa = \kappa$.
So we let $((s_\xi, r_\xi))_{\xi<\kappa}$ be an enumeration of B (with possible repetitions) in such a way that

$$B = \{(s_\xi, r_\xi) : \xi \in I_\beta\} \text{ for each } \ \beta < \kappa$$

For each $J \subseteq \kappa$, we define the function $f_J : \kappa \to \kappa$ as follows :

$$\begin{aligned} f_J(\xi) &= r_\xi(J \cap s_\xi) && \text{if } \ \xi \in \bigcup_{\beta<\kappa} I_\beta \\ f_J(\xi) &= 0 && \text{otherwise} \end{aligned}$$

There are 2^κ subsets J of κ, thus 2^κ functions f_J. We next show that the f_J's are pairwise distinct. Suppose that $J_1 \neq J_2$. By symmetry and without loss of generality, we may suppose that there is an $x \in J_1$ such that $x \notin J_2$. Let $s = \{x\}$ and $r = \{(\{x\}, 0), (\emptyset, 1)\}$. Then $(s, r) \in B$, so $(s, r) = (s_\xi, r_\xi)$ for some ξ. Now $f_{J_1}(\xi) = r(J_1 \cap s) = 0$ and $f_{J_2}(\xi) = r(J_2 \cap s) = 1$; so $f_{J_1} \neq f_{J_2}$.

Now let $\beta, \gamma_1, \ldots, \gamma_n$ be ordinals in κ, and let $J_1, \ldots, J_n$ be distinct subsets of κ. We will show that there is a $\xi \in I_\beta$ such that

$$f_{J_i}(\xi) = \gamma_i \text{ for } 1 \leq i \leq n$$

(note that it will also show that the range of each f_J is κ)
Let $s \in \mathcal{P}_\omega(\kappa)$ be such that $s \cap J_i \neq s \cap J_j$ for $1 \leq i < j \leq n$. It is then possible to define $r : \mathcal{P}(s) \to \kappa$ by $r(J_i \cap s) = \gamma_i$, for $i = 1, \ldots, n$. As there is a $\xi \in I_\beta$ such that $(s_\xi, r_\xi) = (s, r)$, we obtain

$$f_{J_i}(\xi) = r_\xi(J_i \cap s_\xi) = r(J_i \cap s) = \gamma_i$$

We finally put

$$\Pi := \{\{f_J^{-1}(\gamma) : \gamma < \kappa\} : J \subseteq \kappa\}$$

Because each f_J is surjective, and because $f_J^{-1}(\beta) \cap f_J^{-1}(\alpha) = \emptyset$ if $\beta \neq \alpha$, $\{f_J^{-1}(\gamma) : \gamma < \kappa\}$ is a partition of κ ; as there are 2^κ distinct functions f_J, Π is well a collection of 2^κ partitions of κ.

Next let $X \in F$ and $(X_i)_{1 \leq i \leq n}$ such that $X_i \in P_i \in \Pi$ and $P_i \neq P_j$ for $i \neq j$. Then there are γ_i, J_i with $X_i = f_{J_i}^{-1}(\gamma_i)$, and β such that $J_\beta \subseteq X$. Therefore, as shown above, there is $\xi \in I_\beta \subseteq J_\beta$ such that $f_{J_i}(\xi) = \gamma_i$, meaning that

$$X \cap \bigcap_{i=1}^{n} X_i \neq \emptyset$$

i.e. (Π, F) is consistent.

(ii) Suppose that $(\Pi, (F, \{J\}))$ is not consistent. Then there are $X \in F$, $X_i \in P_i \in \Pi$, $1 \leq i \leq n$, the P_i's being pairwise distinct, such that :

$$J \cap X \cap \bigcap_{i=1}^{n} X_i = \emptyset \tag{4.2}$$

Let $\Pi' := \Pi \backslash \{P_1, \ldots, P_n\}$. Let Q_j, $1 \leq j \leq m$, be distinct elements of Π' and $Y_j \in Q_j$. Because $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ are distinct and (Π, F) is consistent, we get :

$$X \cap \bigcap_{i=1}^{n} X_i \cap \bigcap_{j=1}^{m} Y_j \neq \emptyset \tag{4.3}$$

From (4.2) and (4.3) we obtain :

$$(\kappa \setminus J) \cap X \cap \bigcap_{j=1}^{m} Y_j \neq \emptyset$$

So $(\Pi', (F, \{\kappa \setminus J\}))$ is consistent.

(iii) Let X_δ, $\delta < \kappa$ be an enumeration of P without repetition, and, likewise, let $\mathcal{P}_\omega(\kappa) := \{t_\delta : \delta < \kappa\}$. For each $\delta < \kappa$, we define a function $q_\delta : \mathcal{P}_\omega(\kappa) \to \mathcal{P}(\kappa)$ in the following way :

$$\begin{aligned} q_\delta(s) &= p(t_\delta) \cap X_\delta && \text{if } s \subseteq t_\delta \\ q_\delta(s) &= \emptyset && \text{if } s \nsubseteq t_\delta \end{aligned}$$

Note that $s_1 \cup s_2 \subseteq t_\delta$ iff both $s_1 \subseteq t_\delta$ and $s_2 \subseteq t_\delta$, hence $q_\delta(s_1 \cup s_2) = q_\delta(s_1) \cap q_\delta(s_2)$, i.e. the q_δ's are additive functions.

Remark that $q_\delta(s) \subseteq p(t_\delta)$. Since $X_\delta \in P \in \Pi$ and $p(t_\delta) \in F$, consistency of (Π, F) implies that $q_\delta(s) \neq \emptyset$ if $s \subseteq t_\delta$. We next define the function :

$$q : \begin{array}{ccc} \mathcal{P}_\omega(\kappa) & \to & \mathcal{P}(\kappa) \\ s & \mapsto & \bigcup_{\delta<\kappa} q_\delta(s) \end{array}$$

As p is monotonic, we see easily that $q(s) \subseteq p(s)$:

If $x \in q(s)$, there is δ such that $x \in X_\delta \cap p(t_\delta), s \subseteq t_\delta$. Since p is

monotonic, $p(t_\delta) \subseteq p(s)$, i.e. $q \leq p$.
Since $\delta \neq \delta'$ implies $X_\delta \cap X_{\delta'} = \emptyset$ implies $q_\delta(s) \cap q_{\delta'}(s) = \emptyset$, we have that $q(s)$ is a disjoint union of subsets of elements of P.
We can easily see that q is additive using the additivity of q_δ and the fact that $X_\delta \cap X_{\delta'} \neq \emptyset$ iff $\delta = \delta'$.

Now let $F' = (F, \operatorname{Image} q)$. Claim : $(\Pi \setminus \{P\}, F')$ is consistent.
Let $X \in F$, $s \in \mathcal{P}_\omega(\kappa)$, $X_i \in P_i \in \Pi$, $1 \leq i \leq n$, the P_i's being pairwise distinct and different from P. Since $s = t_\delta$ for some $\delta < \kappa$, we have $q(s) \supseteq q_\delta(s) = p(t_\delta) \cap X_\delta$, and

$$X \cap p(t_\delta) \cap X_\delta \cap \bigcap_{1 \leq i \leq n} X_i \neq \emptyset$$

($X_\delta \in P$ so it is distinct from the X_i's. $p(t_\delta) \in F$, so $X \cap p(t_\delta) \in F$. Hence we get the above result by consistency of (Π, F))
Whence

$$X \cap q(s) \cap \bigcap_{1 \leq i \leq n} X_i \neq \emptyset$$

$\square$

We are now able to state and prove the following theorem :

THEOREM 2.25. *Let I be a set of infinite cardinality κ. Then there exists a κ^+-good countably incomplete ultrafilter D over I.*

PROOF. Without loss of generality, we may assume that $I = \kappa$. Let $(I_n)_{n<\omega}$ be a sequence of subsets of κ, each of cardinality κ, such that $I_{n+1} \subseteq I_n$ and $\bigcap_{n<\omega} I_n = \emptyset$ (this is possible because of Proposition 2.17).
Let F_0 be the uniform filter generated by the set $\{I_n : n \in \omega\}$. By Lemma 2.24 (i), let Π_0 be any collection of partitions of κ such that $|\Pi_0| = 2^\kappa$ and (Π_0, F_0) is consistent. We shall define by transfinite induction two sequences $(\Pi_\xi)_{\xi<2^\kappa}$ and $(F_\xi)_{\xi<2^\kappa}$ such that

$$\Pi_\xi \subseteq \Pi_\eta, F_\xi \supseteq F_\eta \text{ if } \eta \leq \xi < 2^\kappa$$

$$|\Pi_\xi| = 2^\kappa, |\Pi_\xi \setminus \Pi_{\xi+1}| < \omega$$

$$\Pi_\lambda = \bigcap_{\eta<\lambda} \Pi_\eta, \lambda \,\text{limit}$$

$$(\Pi_\xi, F_\xi) \text{ is consistent for } \xi < 2^\kappa$$

We make the construction in the following way : Let $(p_\xi)_{\xi<2^\kappa}$ be an enumeration of all monotonic functions mapping $\mathcal{P}_\omega(\kappa)$ into $\mathcal{P}(\kappa)$, and let $(J_\xi)_{\xi<2^\kappa}$, be an enumeration of $\mathcal{P}(\kappa)$.
Suppose that Π_η, F_η for $\eta < \xi < 2^\kappa$ have been defined satisfying all the inductive hypotheses.

- If ξ is a limit ordinal, then simply let

$$\Pi_\xi = \bigcap_{\eta<\xi} \Pi_\eta \text{ and } F_\xi = \bigcup_{\eta<\xi} F_\eta$$

It is clear that $|\Pi_\xi| = 2^\kappa$ and (Π_ξ, F_ξ) is consistent:
let $X \in F_\xi$, and $X_i \in P_i \in \Pi_\xi$, for $i = 1, \ldots, n$ such that the P_i's are pairwise distinct. Then there exists η, $X \in F_\eta$. As $X_i \in P_i \in \Pi_\eta$ for all $i \leq n$, we get

$$X \cap \bigcap_{i=1}^{n} X_i \neq \emptyset$$

- If $\xi = \lambda + 2n + 1$, λ a limit ordinal and $n < \omega$, then let J be the first element of $\mathcal{P}(\kappa)$ not already in $F_{\xi-1}$. By Lemma 2.24 (ii), we can find Π_ξ, F_ξ such that

$$(\Pi_\xi, F_\xi) \text{ is consistent}$$

$$|\Pi_\xi| = 2^\kappa, |\Pi_{\xi-1} \setminus \Pi_\xi| < \omega$$

$$J \in F_\xi \text{ or } (\kappa \setminus J) \in F_\xi$$

- If $\xi = \lambda + 2n + 2$, λ a limit ordinal and $n < \omega$, then let

$$p : \mathcal{P}_\omega(\kappa) \to F_{\xi-1}$$

be the first function in the list $(p_\xi)_{\xi<2^\kappa}$, which we have not already dealt with. By Lemma 2.24 (iii), we can find $\Pi_\xi, F_\xi, q : \mathcal{P}_\omega(\kappa) \to F_\xi$ such that

$$|\Pi_\xi| = 2^\kappa, |\Pi_{\xi-1} \setminus \Pi_\xi| = 1$$

$$q \leq p, q \text{ is additive}$$

$$F_\xi = (F_{\xi-1}, image(q))$$

$$(\Pi_\xi, F_\xi) \text{ is consistent}$$

Let $F = \bigcup_{\xi<2^\kappa} F_\xi$. Because of our construction, we see that F is a countably incomplete κ^+-good ultrafilter over κ. Furthermore, $cf(2^\kappa) > \kappa$ (i.e. within 2^κ there is no unbounded sequence of cardinality less or equal than κ), hence if $p : \mathcal{P}_\omega(\kappa) \to F$, $|domain(p)| = \kappa$, thus there is $\xi < 2^\kappa$ such that $image(p) \subseteq F_\xi$, and therefore the previous construction shows that there is an additive function q refining p.

□

4.3. Main theorem.

THEOREM 2.26. *Let κ be an infinite cardinal and let D be a countably incomplete κ-good ultrafilter over a set I. Suppose $|\mathcal{L}| < \kappa$. Then for any family $(\mathcal{M}_i)_{i\in I}$ of models for $\mathcal{L}$, the ultraproduct $\Pi_D\mathcal{M}_i$ is κ-saturated.*

Proof. It is sufficient to prove that for every set $\Sigma(x)$ of formulas of $\mathcal{L}$, if every finite subset of $\Sigma(x)$ is satisfiable in $\Pi_D \mathcal{M}_i$, then $\Sigma(x)$ is satisfiable in $\Pi_D \mathcal{M}_i$.

Suppose that every finite subset of $\Sigma(x)$ is satisfiable in $\Pi_D \mathcal{M}_i$. As D is countably incomplete, by definition there is a descending chain

$$I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

such that each $I_n \in D$ and $\bigcap_{n<\omega} I_n = 0$. We also know that $|\Sigma| < \kappa$ because $|\mathcal{L}| < \kappa$ and Σ is a countable union of subsets of $\mathcal{P}_\omega(\mathcal{L})$ (Σ can be written as a union, indexed by $\mathbb{N}$, of sets of formulas, the length of which is less or equal to $n \in \mathbb{N}$. It is possible to encode such a formula (and thus, such a set of formulas) by a subset of $\mathcal{P}_\omega(\mathcal{L})$, if we extend $\mathcal{L}$ with symbols representing $\forall, \exists, (,)$ and with numbers to quote the position of a given symbol in the formula). Let $f : \mathcal{P}_\omega(\Sigma) \to D$ be such that for every finite subset σ of Σ,

$$f(\sigma) = I_{|\sigma|} \cap \{i \in I : \mathcal{M}_i \vDash \exists x \bigwedge \sigma\}$$

with the understanding that $f(\emptyset) = I$. Each $\sigma \in \mathcal{P}_\omega(\Sigma)$ is finite and is satisfiable in $\Pi_D \mathcal{M}_i$, whence $\Pi_D \mathcal{M}_i \vDash \exists x \bigwedge \sigma$. By the Transfer Principle 2.13, $f(\sigma) \in D$.

Let $\sigma \subseteq \tau \in \mathcal{P}_\omega(\Sigma)$. Then

$$I_{|\tau|} \subseteq I_{|\sigma|} \text{ and } (\exists x \bigwedge \tau \to \exists x \bigwedge \sigma)$$

so $f(\tau) \subseteq f(\sigma)$ i.e. f is monotonic.

Since D is κ-good, there is an additive function $g \leq f$ on $\mathcal{P}_\omega(\Sigma)$ into D. For each $i \in I$, let

$$\sigma(i) := \bigcup \{\theta \in \Sigma : i \in g(\{\theta\})\}$$

Claim : if $|\sigma(i)| \geq n$, then $i \in I_n$.
Indeed if $\sigma(i)$ has at least n distinct elements $\theta_1, \ldots, \theta_n$, then for $s \leq n$ we have $i \in g(\theta_s)$, whence using the additivity of g :

$$i \in g(\{\theta_1\}) \cap \ldots \cap g(\{\theta_n\}) = g(\{\theta_1, \ldots, \theta_n\}) \subseteq f(\{\theta_1, \ldots, \theta_n\}) \subseteq I_n$$

As $\bigcap_{n<\omega} I_n = \emptyset$, for each $i \in I$, $\sigma(i)$ is finite.

Then we pick an element h_D which satisfies $\Sigma(x)$ in $\Pi_D \mathcal{M}_i$. For each $i \in I$, we have by definition of $f(\sigma)$ and of $\sigma(i)$, and by additivity :

$$i \in \bigcap \{g(\{\theta\}) : \theta \in \sigma(i)\} = g(\sigma(i)) \subseteq f(\sigma(i))$$

so $i \in f(\sigma(i))$.

Next, by construction of $f(\sigma)$ it is possible to choose an element $h(i) \in M_i$ for which $\bigwedge \sigma(i)$. That can also be written

$$\mathcal{M}_i \models \bigwedge \sigma(i)\,[h(i)]$$

Henceforth, whenever $\theta \in \sigma$ and $i \in g(\{\theta\})$, we have $\theta \in \sigma(i)$ and then $\mathcal{M}_i \models \theta\,[h(i)]$. But $g(\{\theta\}) \in D$, so by the Transfer Principle 2.13 $\Pi_D \mathcal{M}_i \models \theta\,[h_D]$ for all $\theta \in \Sigma$, which shows that h_D satisfies Σ in $\Pi_D \mathcal{M}_i$. $\square$

CHAPTER 3

The nonstandard setting

Let $G := (G, 1, ., ^{-1})$ be a local group as defined in 1.12, with the extra assumptions made in Section 4 of Chapter 1, namely $\Lambda = G$ and we suppose that G has been restricted to U_3; hence G is a homogeneous space in which the inversion and involution laws are satisfied:

- If $(x, y) \in \Omega$ and $xy = 1$, then $x = y^{-1}$ and $y = x^{-1}$. Thus, for all $x \in G$, $(x^{-1})^{-1} = x$.
- If $(g, h) \in \Omega$ then $(h^{-1}, g^{-1}) \in \Omega$ and $(gh)^{-1} = h^{-1}g^{-1}$.

Set $\mathcal{L} := \{1, ., ^{-1}\}$. Let $\mathcal{O} := (O_i)_{i \in I}$ be a base of open neighborhoods of the identity in G. In order to express properties of the local group G within first order logic, we extend $\mathcal{L}$ into a language $\mathcal{L}_P := \mathcal{L} \cup \{P_i, i \in I\}$, where P_i is a unary relation symbol, interpreted in G by the open set O_i. We can thus express a statement such as "$x \in O_i$" by the formula $P_i(x)$. In other words, $P_i(G) = O_i$, i.e. the set O_i is definable in the language $\mathcal{L}_P$.

Note that we will abuse notation and identify the local group G and the $\mathcal{L}_P$-structure $(G, 1, ., ^{-1}, (P_i)_{i \in I})$. Under our assumptions, for $i \in I$ and $g \in G$, $\{g\} \times O_i \subseteq \Omega$ and $O_i \times \{g\} \subseteq \Omega$, then $g.O_i$ and similarly $O_i.g$ are open neighborhoods of g by Lemma 1.14, and O_i^{-1} is an open neighborhood of 1. For $g \in G$, we let $\mathcal{O}_g$ denote the set of open neighborhoods of g.

Next, to express properties of elements lying in a "small" neighborhood of 1, we will consider a saturated elementary extension of G in which we will have "infinitesimal" elements. Let $\mathcal{U}$ be an ultrafilter on I which is $|I|^+$-good and countably incomplete. This is possible through Theorem 2.25. The ultrapower of G, namely $G^* := \prod_{\mathcal{U}} G$ is then $|I|^+$-saturated by Theorem 2.26; and it is also an elementary extension of G by Corollary 2.14. We let $\kappa := |I|^+$.

This section is mainly based on *Nonstandard Analysis* [**19**] and *An invitation to Nonstandard analysis* [**11**], respectively written by A.Robinson and T.Lindstrøm. However, unlike their approaches , we stay in ultrapowers of first-order structures. Consequently, we have adapted proofs and some notions in a slightly different manner for the presentation here. The section is also based on [**5**].

1. Internal sets and general properties

We begin with general properties, most of which stay true (considering a little adaptation of the proof) if G is a topological space which is not especially a local group. We also mention some properties of $\mathbb{R}^*$ and $\mathbb{N}^*$, ultrapowers of the structures $(\mathbb{R}, +, ., <, 0, 1)$ and $(\mathbb{N}, +, <, 0)$.

For the sake of readability, an element of G^* will be denoted x or $x = \langle x_i \rangle$ when accent is put on its "mathematical nature" of an "equivalence class of elements of an infinite cartesian product". Whenever it is possible, a similar notation is used for sets and functions; this restriction will be explained below. Writing $x_k = \langle x_{k,i} \rangle$, the Transfer Principle can be expressed as follows :
If $\phi(v_1, \dots, v_n)$ is a $\mathcal{L}_P$-formula,

$$G^* \vDash \phi(x_1, \dots, x_n) \text{ iff } \{i \in I : G \vDash \phi(x_{1,i}, \dots, x_{n,i})\} \in \mathcal{U}$$

Let $(A_i)_{i \in I}$ be a sequence of subsets of G. We can add unary predicates R_i to the language, such that $R_i^G(x)$ iff $x \in A_i$. Note that the cardinality of the expanded language is still $|I|$. We then define a subset $\langle A_i \rangle$ of G^* by

$$\langle x_i \rangle \in \langle A_i \rangle \text{ iff } \{i : R_i^G(x_i)\} \in \mathcal{U}$$

Definition 3.1. (1) *A sequence $(A_i)_{i \in I}$ of subsets of G defines a subset $\langle A_i \rangle$ of G^* by*

$$\langle x_i \rangle \in \langle A_i \rangle \text{ iff } \{i : x_i \in A_i\} \in \mathcal{U}$$

A subset of G^* which can be obtained in this way is called ***internal.***

(2) *A sequence $(f_i)_{i \in I}$ of functions from G to G defines a function $\langle f_i \rangle$ from G^* to G^* by*

$$\langle f_i \rangle (\langle x_i \rangle) = \langle f_i(x_i) \rangle$$

Any function that can be obtained in this way is called ***internal.***

For instance, we can consider the internal function $f^* = \langle f, f, f, \dots \rangle$, or, for each $A \subseteq G$, the internal set $A^* = \langle A, A, A, \dots \rangle$.

Let $x = \langle x_i \rangle$ and $O_j \in \mathcal{O}$.
Then $\{i \in I : P_j(x_i)\} \in \mathcal{U}$ iff $x \in P_j(G^*) = O_j^*$.

Notice also that the family of internal sets is closed under finite Boolean operations, and that the "product-like" structure of internal sets allows to lift standard properties componentwise.

Definition 3.2. *An element $x \in \mathbb{R}^*$ is* ***finite*** *if $-a < x < a$ for some positive real number a. An element which is not finite is called* ***infinite.***

For example, if $I = \omega$, two infinite numbers are $\langle x_n \rangle := \langle n \rangle$ and $\langle y_n \rangle := \langle -n^2 \rangle$. Notice that there is no smallest infinite number. If $\nu \in \mathbb{N}^* \setminus \mathbb{N}$, i.e.

ν is an infinite number of $\mathbb{N}^*$, we write $\nu > \mathbb{N}$.

Proofs of Propositions 3.3 and 3.4 are those from [**11**]. We include them for completeness.

PROPOSITION 3.3. *An internal, non-empty subset of* $\mathbb{R}^*$ *which is bounded above has a least upper bound.*

PROOF. [**11**] If the internal set $A = \langle A_i \rangle$ is bounded above by $a = \langle a_i \rangle$, then the set of i's such that the corresponding A_i's are bounded by a_i's belongs to the ultrafilter $\mathcal{U}$. Hence we may assume without loss of generality that all the A_i's are bounded above. Then $b = \langle supA_i \rangle$ is the least upper bound of A, i.e. $b = sup\,\langle A_i \rangle$. □

PROPOSITION 3.4. *Let A be an internal subset of* $\mathbb{R}^*$.

Overflow *If A contains arbitrarily large finite elements, then A contains an infinite element.*

Underflow *If A contains arbitrarily small positive infinite elements, then A contains a finite element.*

PROOF. [**11**] Overflow: If A is unbounded, there is nothing to prove. Thus let a be A's least upper bound; a is clearly infinite, and there must be an $x \in A$ such that $\frac{a}{2} \leq x \leq a$.
Underflow: Let b be the greatest lower bound of the set A^+ of positive elements in A; then b is finite, and there must be an $x \in A$ such that $b \leq x \leq b + 1$. □

Recall that a set is said to be well-ordered if it is equipped with an order such that there is no infinite decreasing sequence of elements. The nonstandard natural numbers are not well-ordered, which does not contradict the Transfer Principle, but which shows that the property of being well-ordered cannot be expressed within first-order logic.

PROPOSITION 3.5 (**Internal induction**). *If $A \subseteq \mathbb{N}^*$ is internal, contains* 0 *and is closed under the successor operation, then $A = \mathbb{N}^*$.*

PROOF. It is an application of the Transfer Principle. □

DEFINITION 3.6. *An **internal (hyper)finite set** is an internal set $A = \langle A_i \rangle$ where*

$$\{i \in I : A_i \text{ is finite}\} \in \mathcal{U}$$

A lot of properties of finite sets can be extended to hyperfinite internal sets.
Another definition is that an internal set $A = \langle A_i \rangle$ is (hyper)finite iff there is $\nu \in \mathbb{N}^*$ and an internal bijection $f : \{1, \ldots, \nu\} \to A$. This definition is easily seen to be equivalent, using the notion of cardinality : $|A| := \langle |A_i| \rangle$.

DEFINITION 3.7. *Let $\nu = \langle \nu_i \rangle \in \mathbb{N}^*$ (finite or infinite), and let A be the internal (hyper)finite set $\langle \{n \in \mathbb{N} : 1 \leq n \leq \nu_i\} \rangle$. A **internal (hyper)finite***

sequence $(x_k)_{k\in A}$ *of elements of* G^* *is an internal function* $k \mapsto x_k$ *from* A *to* G^**, i.e.* $(x_k)_{k\in A} = (\langle x_i(k_i)\rangle)_{k\in A}$.

It can also be seen in the following way, using a formalization which is closer to the definition of a cartesian product : let $(f_n)_{n\in\omega}$ be a sequence of functions $f_n : I \to G$, and $\nu = \langle \nu_i \rangle$. For $k \in A$, the element f_k of G^* is the equivalence class of the sequence $(f_{k_i}(i))_{i\in I}$.

PROPOSITION 3.8 (**Internal definition principle**). *If* B *is an internal set, and* $a_1, \ldots, a_n$ *elements in* $\prod_{\mathcal{U}} G$ *and* ψ *is a formula in the language* $\mathcal{L}_P$*, then*

$$D = \{c \in B : \psi(c, a_1, \ldots, a_n)\}$$

is an internal set.

PROOF. Write $B = \langle B_i \rangle$ and $\overline{a} := (a_1, \ldots, a_n)$. Let R be a symbol of a unary predicate, which we add to $\mathcal{L}_P$. For $i \in I$, we define structures $G_i := (G, B_i)$, the domain of which is G and in which the predicate R is interpreted by $R(G_i) = B_i$, or equivalently: for $x \in G_i$, $R^{G_i}(x)$ iff $x \in B_i$. Next we consider the ultraproduct of the G_i's (instead of considering an ultrapower of G). Then

$$\prod_{\mathcal{U}} G_i \vDash (\psi(c, \overline{a}) \wedge R(c)) \text{ iff } \{i \in I : G_i \vDash \psi(c_i, \overline{a_i}) \wedge R(c_i)\} \in \mathcal{U}$$

Hence

$$D = \langle \{c_i \in B_i : \psi(c_i, \overline{a_i})\} \rangle$$

□

PROPOSITION 3.9. *An infinite, internal set in a* κ*-saturated ultraproduct of* G *has cardinality at least* κ.

PROOF. Let A be an infinite internal set, the cardinality of which, namely α, is strictly less than κ, i.e. less or equal to $|I|$. We can write $A = \langle A_i \rangle$. As

$$\{i \in I : A_i \text{ is infinite}\} \in \mathcal{U}$$

we can assume that A_i is infinite for all $i \in I$, and, further, of cardinality α without loss of generality. Next we extend the language $\mathcal{L}_P$ with symbols c_δ, $\delta < \alpha$, and with a unary predicate symbol R. Note that the cardinality of the extended language is still $\leq |I|$. We then define, for each $i \in I$, a structure G_i in which

$$\forall x \;\; R^{G_i}(x) \text{ iff } x \in A_i$$

The symbols c_δ, $\delta < \alpha$, are interpreted in G_i as an enumeration without repetition of elements of A_i, and likewise by elements of A in the ultraproduct. We next consider the family of sets $\{A \setminus \{a\}\}_{a\in A}$, each of which can now be written $\langle A_i \setminus \{a_i\}\rangle$ by Proposition 3.8. In G_i, a finite intersection of sets of the form $\bigcap_{k=1}^{n} A_i \setminus \{a_{k,i}\}$ is nonempty because A_i is infinite. Hence the type

$$\{x : R(x) \wedge (\exists \delta\ \delta < \alpha \wedge x \neq c_\delta) \wedge \bigwedge_{\gamma,\delta<\alpha,\gamma\neq\delta} c_\gamma \neq c_\delta\}$$

is satisfiable. Using κ-saturation in the ultraproduct $\prod_{\mathcal{U}} G_i$, we get

$$\bigcap_{a\in A} (A \setminus \{a\}) \neq \emptyset$$

which is a contradiction. □

This shows in particular that if we are working with a set S containing $\mathbb{R}$ and a topological space with a base of cardinality κ, then $\mathbb{R}^*$ has cardinality at least κ within the nonstandard κ-saturated model S^*. That means it is impossible to fix a canonical set $\mathbb{R}^*$ once and for all.

Definition 3.10 (**monad**). *The monad of a point $p \in G$, denoted by $\mu(p)$ is*

$$\mu(p) := \bigcap_{O\in\mathcal{O}_p} O^*$$

In particular, $\mu(1) = \bigcap_{i\in I} O_i^*$.

Note that $\mu(p) = p.\mu(1)$.
First note that $p.\mu(1)$ is well defined: let $J := \{i \in I : \{p\} \times O_i \subseteq \Omega\}$, it is nonempty by Lemma 1.14. As in the proof of Lemma 1.14, let $\Omega_p := \{h \in G : (p,h) \in \Omega\}$. It is an open set containing 1. Then $\mu(1) = \bigcap_{i\in I} O_i^* = \bigcap_{i\in J} O_i^* \cap \bigcap_{i\in I\setminus J}(\Omega_p^* \cap O_i^*)$. Hence we can consider $p.\mu(1)$.
Proof of the equality: if $x \in \mu(p)$, then x belongs to all standard open sets containing p, in particular x belongs to $p.O_i^*$, for $i \in J$, and to $p.(\Omega_p^* \cap O_i^*)$, for $i \in I\setminus J$. Now suppose x belongs to $p.\mu(1)$. Let W be an open neighborhood of p in G. By continuity of the multiplication, there is V a neighborhood of p and O_i such that $V \times O_i \subseteq \Omega$ and $V.O_i \subseteq W$, i.e. $x \in p.O_i \subseteq W$, whence $x \in \mu(p)$. (Note that $p \in G$, and we also denote $\langle p,p,p,\ldots\rangle$ by p).
It also shows that $(p.O_i)_{i\in J}$ is a base of neighborhoods of p.

The monad of a point p is not necessarily an internal set. If p is not isolated, the type $\{P_i(x) \wedge x \neq p : i \in I\}$ is satisfiable since it is finitely satisfiable (a finite intersection of $O_i \setminus \{p\}$'s is nonempty in G). It is realized in G^* by κ-saturation. Thus the monad can be seen as the realization of a type, and $\mu(p) \setminus \{p\}$ is nonempty.

Next we transfer some standard properties in the ultrapower.

Theorem 3.11. *Let $A \subseteq G$, and $x \in G$*
(i) x is in the interior $int(A)$ of A iff $\mu(x) \subseteq A^*$
(ii) x is in the closure $\bar{A}$ of A iff $\mu(x) \cap A^* \neq \emptyset$

PROOF. Let $A \subseteq G$. We consider a unary relation symbol R, which we add in the language, and which is interpreted in G by $R^G(x)$ iff $x \in A$.

(i) $\Rightarrow$: Let $x \in int(A)$: there is an open set $x.O_i \subseteq A$. By transfer we get $(x.O_i)^* \subseteq A^*$. As $\mu(x) \subseteq (x.O_i)^*$, it is then included in A^*.
$\Leftarrow$: If x is not in the interior of A, then for all open sets containing x, $V \cap A^c \neq \emptyset$. In particular, for all $i \in I$, $x.(O_i \cap \Omega_x) \cap A^c \neq \emptyset$. We add a predicate for Ω_x in the language. By transfer we get that for all $i \in I$, $(x.(O_i \cap \Omega_x))^* \cap (A^c)^* \neq \emptyset$. As this stays true through finite intersections of $x.(O_j \cap \Omega_x)$'s, the type $\{y \in x.(O_i \cap \Omega_x) : i \in I\}$ is finitely satisfiable hence satisfiable by the Compactness Theorem 2.7, thus by κ-saturation we get $\mu(x) \cap (A^c)^* \neq \emptyset$, i.e. $\mu(x) \nsubseteq A^*$.

(ii) Apply (i) to A^c.

$\square$

DEFINITION 3.12 (**nearstandard**). *An element $x \in G^*$ is called* ***near-standard*** *(abbreviated ns) if it belongs to $\mu(p)$ for some $p \in G$. The set of nearstandard points in G^* is denoted G^*_{ns}*

$$G^*_{ns} = \bigcup_{g \in G} g.\mu(1)$$

In case G is T_2, monads of distinct elements are disjoint, hence an element $x \in G^*_{ns}$ is then nearstandard to exactly one element $p \in G$. It is then possible to define the **standard part** of x by $st(x) := p$. We can think of it as a map $st : G^*_{ns} \to G$.
We say that x and y are **infinitely close** if they are nearstandard and have the same standard part. This is denoted by $x \sim y$.

If G is T_2, Theorem 3.11 can be rephrased in the following way:

THEOREM 3.13. *(i') A is open in G iff $st^{-1}(A) \subseteq A^*$*
(ii') A is closed in G iff $A^* \cap G^*_{ns} \subseteq st^{-1}(A)$

PROPOSITION 3.14. *If G is T_2 and A is an internal subset of G^*, then $st(A)$ is closed.*

PROOF. As an internal set, A can be written $A = \langle A_i \rangle$, and we can suppose that each $A_i = st(A)$. Once again, we add a predicate R to $\mathcal{L}_P$ interpreted in G as follows : $\forall x \; R^G(x)$ iff $x \in st(A)$.

Let $a \in \overline{st(A)}$. Then, by definition of the closure of a set: for all $O \in \mathcal{O}_a$, we have that $O \cap st(A) \neq \emptyset$. In particular for all $i \in I$, $a.(O_i \cap \Omega_a) \cap st(A) \neq \emptyset$. By transfer the family $((a.(O_i \cap \Omega_a))^* \cap A)_{i \in I}$ has the finite intersection property, and by κ-saturation, the set

$$\bigcap_{i \in I} ((a.(O_i \cap \Omega_a))^* \cap A)$$

has an element x. As x then belongs to the monad of a, it is obviously an element of A such that $st(x) = a$. Thus $a \in st(A)$, so we have shown that $\overline{st(A)} \subseteq st(A)$, i.e. that $st(A)$ is closed. $\square$

THEOREM 3.15. *$A \subseteq G$ is compact iff $A^* \subseteq G^*_{ns}$.*
In other words, $A \subseteq G$ is compact iff all points in A^* are nearstandard iff for every point $q \in A^*$, there is a standard point $p \in A$ such that $q \in \mu(p)$.

PROOF. In this proof, we assume that I is the set of indices for a base of a topology, and not only for the base of neighborhoods of the identity. As previously, we add a symbol R of a unary predicate to $\mathcal{L}_P$ such that $R^G(x)$ iff $x \in A$.

$\Rightarrow$: Suppose that A is compact and that there is a point p in $A^* \setminus G^*_{ns}$. Then p is not contained in the monad of any (standard) point in A nor in G. Thus, every point $x \in A$ possesses an open neighborhood $x.O_i$ such that $p \notin (x.O_i)^*$. Let $\{x.O_i : x \in A\}$ be an open covering of A. As A is compact we can extract a finite subcovering $\{x.O_1, \ldots, x.O_k\}$, say, $k \geq 1$, such that

$$x.O_1 \cup \ldots \cup x.O_k = A$$

This formula of $\mathcal{L}_P$, interpreted in G^*, yields

$$(x.O_1)^* \cup \ldots \cup (x.O_k)^* = A^*$$

which entails that $p \in (x.O_j)^*$ for some j, $1 \leq j \leq k$, which is a contradiction.

$\Leftarrow$: Suppose A is not compact; then there is a family of closed sets, which can be indexed by I, $\{F_i\}_{i \in I}$ such that $\{F_i \cap A\}_{i \in I}$ has the finite intersection property, but

$$\bigcap_{i \in I} (F_i \cap A) = \emptyset$$

In G^* the family $\{F_i^* \cap A^*\}_{i \in I}$ also has the finite intersection property, but by $|I|^+$-saturation $\bigcap_{i \in I} (F_i^* \cap A^*)$ must have an element x. Assume that x is nearstandard to an element a in A, i.e. that there is $a \in A$, $x \in \mu(a)$. Then $\mu(a) \cap F_i^* \neq \emptyset$ because it contains x. Since F_i is closed, it follows from Theorem 3.11 that $a \in F_i$ for all $i \in I$. But then, by Transfer, $a \in \bigcap_{i \in I} (F_i \cap A)$ in G, which is a contradiction. Hence x cannot be nearstandard to an element of A, and the proof is complete. $\square$

Note that when $A \subseteq X$ is compact, then $A^* \subseteq A^*_{ns}$.

COROLLARY 3.16. *A topological group or a local group G is locally compact if there is an open set U containing the identity and such that $\overline{U}^* \subseteq G^*_{ns}$.*

PROOF. Lemma 1.14 and Theorem 3.15 $\square$

PROPOSITION 3.17. *A function $f : X \to Y$ is continuous at $a \in X$ if and only if*

$$f^*(\mu(a)) \subseteq \mu(f(a))$$

PROOF.

$\Rightarrow$: let f be continuous at a and $x \in \mu(a)$. Let $O := f(a).O_i$ be a neighborhood of $f(a)$. The aim is to show that $f^*(x) \in O^*$. By continuity of f, there is a neighborhood $V \in \mathcal{O}_a$ of a such that $f(V) \subseteq O$, so $f^*(V^*) \subseteq O^*$. (Writing $f^* = \langle f, f, f, \ldots \rangle$, $V^* = \langle V, V, V, \ldots \rangle$, we see that

$$f^*(V^*) = \langle f(V), f(V), f(V), \ldots \rangle$$

and each $f(V) \subseteq O$.) As $x \in V^*$, it follows that $f^*(x) \in O^*$.

$\Leftarrow$: let $J := \{i \in I : \{a\} \times O_i \subseteq \Omega\}$. We add a predicate in the language corresponding to $\Omega_a = \{g \in G : (a, g) \in \Omega\}$. Suppose that f is not continuous at a. There is then a neighborhood $O := f(a).O_j$ of $f(a)$ such that

$$\text{for } i \in J, A_i = \{x : x \in a.O_i \text{ and } f(x) \notin f(a).O_j\} \neq \emptyset$$

and

$$\text{for } i \in I \setminus J, A_i = \{x : x \in a.(O_i \cap \Omega_a) \text{ and } f(x) \notin f(a).O_j\} \neq \emptyset$$

The family $\{A_i^*\}_{i \in I}$ has the finite intersection property, and by κ-saturation the sets A_i^* have a common element u. By definition $u \in \mu(a)$, but $f^*(u) \notin (f(a).O_j)^*$. ☐

2. Infinitesimals in a local group

Let G be a local group. An element which belongs to the monad of the identity $\mu(1)$ is called an **infinitesimal**. For example, within the hyperreals: an element $x \in \mathbb{R}^*$ is **infinitesimal** if $-a < x < a$ for all positive real numbers a. Note that zero is the only infinitesimal real number. Other infinitesimals are, for an example if $I = \omega$, $\langle 1/n \rangle$ and $\langle 1/\sqrt{n} \rangle$;

$$\mu(0) = \bigcap_{n \in \mathbb{N} \setminus \{0\}} \left] -\frac{1}{n}; \frac{1}{n} \right[^*$$

as $\mathbb{R}$ admits a countable basis for the usual topology, and, in particular, a countable basis of neighborhoods of 0, namely

$$\left(\left] -\frac{1}{n}; \frac{1}{n} \right[\right)_{n \in \mathbb{N} \setminus \{0\}}$$

LEMMA 3.18. *(1) Let $a, b \in G$, such that $(a, b) \in \Omega$. Then $\mu(a)\mu(b) \subseteq \mu(ab)$ in G^*.*

(2) Let $a \in G$, such that $a \in \Lambda$. Then $\mu(a^{-1}) = (\mu(a))^{-1}$ in G^*.

PROOF. (1) Let $a' \in \mu(a)$ and $b' \in \mu(b)$, and let $W = ab.O_k$ be an open neighborhood of ab in G. By continuity of p on Ω, there are open sets $U = a.O_i$, $V = b.O_j$ in G such that $U \times V \subseteq \Omega$, $a \in U$, $b \in V$, and $UV \subseteq W$. Then $a' \in U^*$, $b' \in V^*$, and $U^*V^* \subseteq W^*$. Hence $a'b' \in W^*$ for any neighborhood W of ab in G, i.e. $a'b' \in \mu(ab)$.

(2) The proof is similar : let $a' \in \mu(a)$ and let V be an open neighborhood of a^{-1} in G. By continuity of ι, there is an open neighborhood U of a in G, such that $U \subseteq \Lambda$ and $U^{-1} \subseteq V$. Hence $(a')^{-1} \in V^*$, i.e. $(\mu(a))^{-1} \subseteq \mu(a^{-1})$. Now if we apply the same argument to a^{-1} instead of a, we get $(\mu(a^{-1}))^{-1} \subseteq \mu(a)$, so $\mu(a^{-1}) \subseteq (\mu(a))^{-1}$ and thus $(\mu(a))^{-1} = \mu(a^{-1})$.

□

THEOREM 3.19. *$\mu := \mu(1)$ is a normal subgroup of G^*.*

PROOF. Let $a, b \in \mu(1)$. By the Lemma 3.18, we have that $ab \in \mu(1)\mu(1) = \mu(1)$ and that $a^{-1} \in \mu(1^{-1}) = \mu(1)$. This proves that μ is a subgroup of G^*. The fact that it is also normal is shown similarly using continuity of the conjugation. □

DEFINITION 3.20. *$\mu := \mu(1)$ is called the **infinitesimal group** of G^*.*

LEMMA 3.21. *(1) suppose $a, b \in G$, $a' \in \mu(a)$ and $b' \in \mu(b)$. If $(a, b) \in \Omega$, then $(a', b') \in \Omega^*$, $a'.b' \in G^*_{ns}$, and $st(a'.b') = a.b$.*

(2) For any $a \in G^*_{ns}$ and $b \in \mu$, $(a, b), (b, a) \in \Omega^*$, $a.b, b.a \in G^*_{ns}$, and $st(a.b) = st(b.a) = st(a)$

(3) For any $a, b \in G^*_{ns}$, if $(a, b^{-1}) \in \Omega^*$ and $a.b^{-1} \in \mu$, then $a \sim b$.

(4) For any $a \in G$, $a' \in \mu(a)$, and any n, if a^n is defined, then $(a')^n$ is defined and $(a')^n \in \mu(a^n)$.

The proof is immediate using Lemma 3.18.

LEMMA 3.22. *Suppose U is a neighborhood of 1 in G and $a \in \mu$. Then there is $\nu > \mathbb{N}$ such that a^σ is defined and $a^\sigma \in U^*$ for all $\sigma \in \{1, \dots, \nu\}$.*

PROOF. Add a predicate R in the language, such that $R^G(x)$ iff $x \in U$.Let $X := \{\sigma \in \mathbb{N}^* : a^\sigma \text{ is defined and } a^\sigma \in U^*\}$. Then X is an internal subset of $\mathbb{N}^*$ which contains $\mathbb{N}$ since μ is a subgroup of G^*. Hence by overflow (Proposition 3.4), there is a $\nu > \mathbb{N}$ such that $\{0, 1, \dots, \nu\} \subseteq X$. □

Let $a_1, \dots, a_\nu$ be an internal sequence of elements of G^* with $\nu > 0$, $\nu = \langle \nu_i \rangle$. We want to define the product $a_1 \dots a_k$ for all $k \in \{1, \dots, \nu\}$. The idea is to construct the product componentwise in G, i.e. (1) for every $i \in I$, the element $a_1(i). \cdots .a_{k_i}(i)$ of G, and then to consider (2) the element $\langle a_1(i). \cdots .a_{k_i}(i) \rangle$ of G^*.
(1): for $i \in I$, the element $a_1(i). \cdots .a_{k_i}(i)$ of G is defined if it follows the rules stated in Definition 1.24.
(2): we show that this does not depend on the chosen representative. Let

$(m_i)_{i\in I}$ be such that $\langle m_i\rangle = k$. As $\{i \in I : m_i = k_i\} \in \mathcal{U}$, we have that $\{i \in I : a_1(i).\cdots.a_{m_i}(i) = a_1(i).\cdots.a_{k_i}(i)\} \in \mathcal{U}$.

LEMMA 3.23. *Suppose G is locally compact. Let $a_1, \ldots, a_\nu$ be an internal sequence of elements of G^* with $\nu > 0$ such that for all $i \in \{1, \ldots, \nu\}$ we have $a_i \in \mu$, $a_1 \ldots a_i$ is defined and $a_1 \ldots a_i \in G^*_{ns}$. Then the set*

$$S := \{st(a_1 \ldots a_i) : 1 \leq i \leq \nu\} \subseteq G$$

is compact and connected (and contains 1).

PROOF. First note that $st(a_1) = 1$, as $a_1 \in \mu$. Next, to prove S is compact, we show that S is a closed subset of a compact.
By Proposition 3.14, S, as the standard part of an internal set, must be closed.
Now suppose towards a contradiction that for every compact $K \subseteq G$, $S \nsubseteq K$. As K^* is an internal set of G^*, there is $k \in \{1, \ldots, \nu\}$ such that $a_1 \ldots a_k \notin K^*$; i.e. $\langle a_1(i) \ldots a_{k_i}(i)\rangle \notin K^*$, i.e. $\{i \in I : a_1(i) \ldots a_{k_i}(i) \notin K\} \in \mathcal{U}$. We suppose that K contains 1, and, without loss of generality, that $a_1(i) = 1$. Then, for each $i \in I$, let $l_i \in \{1, \ldots, k_i\}$ be the smallest element such that $a_1(i) \ldots a_{l_i}(i) \notin K$. Thus $a_1(i) \ldots a_{l_i-1}(i) \in K$, i.e. $a_1 \ldots a_l \in K^*$, but $st(a_1 \ldots a_{l-1}).st(a_l) = st(a_1 \ldots a_{l-1}).1 = st(a_1 \ldots a_l)$, yielding a contradiction. Hence there is a compact $C \subseteq G$ with $a_1 \ldots a_i \in C^*$ for all $i \in \{1, \ldots, \nu\}$, so $S \subseteq C$.

Now suppose S is not connected. Then we have disjoint open subsets U and V of G such that $S \subseteq U \cup V$ and $S \cap U \neq \emptyset$, $S \cap V \neq \emptyset$. Assume $1 \in U$. Then, as $a_1 \in \mu$, $a_1 \in U^*$. Since $S \cap V \neq \emptyset$, $S^* \cap V^* \neq \emptyset$, so we can choose $i \in \{1, \ldots, \nu\}$ minimal such that $a_1 \ldots a_i \in V^*$. Then $i \geq 2$ and $a_1 \ldots a_{i-1} \in U^*$. Now using Lemma 3.21 item (2), we have $a := st(a_1 \ldots a_{i-1}) = st(a_1 \ldots a_i) \in S$, since $a_i \in \mu$ and so $st(a_i) = 1$. If $a \in U$, then $a_1 \ldots a_i \in U^*$ and if $a \in V$, then $a_1 \ldots a_{i-1} \in V^*$: both options lead to a contradiction. $\square$

CHAPTER 4

Growth of powers of infinitesimals

In this chapter, which is mainly based on [**5**], we examine different ways powers of infinitesimals can grow. In particular the notion of purity is defined; some links with the NSS and NSCS properties are shown. From now on, we will not mention explicitly in the proofs that it is necessary to add predicates in the language to have definable sets to use saturation.

1. Asymptotic notations, pure infinitesimals

Let $\nu, \sigma, \tau, \eta, N$ range over $\mathbb{N}^*$, i and j range over $\mathbb{Z}^*$, m and n range over $\mathbb{N}$. Let $x, y \in \mathbb{R}^*$.
Recall the Landau notations :

- We say that $x = o(y)$ if for all $n > 0$, $|x| < \frac{y}{n}$
- We say that $x = O(y)$ if there is some $n > 0$ for which $|x| < ny$

Hence x is infinitesimal iff $x = o(1)$, and x is finite iff $x = O(1)$. For $y \neq 0$, $x = O(y)$ iff $\frac{x}{y}$ is finite, and $x = o(y)$ iff $\frac{x}{y}$ is infinitesimal. The elements x and y are in the same "archimedean class" if $x = O(y)$ and $x \neq o(y)$.

From now on, we let the local group G be locally compact. Considering elements in μ, we want to define a kind of "order of infinitesimality", by describing the different ways their powers can grow:

Definition 4.1. *Let $\nu \in \mathbb{N}^*$ such that $\nu > \mathbb{N}$. Let*

$$G(\nu) := \{a \in \mu : a^i \textit{ is defined and } a^i \in \mu \textit{ for all } i = o(\nu)\}$$

$$G^o(\nu) := \{a \in \mu : a^i \textit{ is defined and } a^i \in \mu \textit{ for all } i = O(\nu)\}$$

Note that the former contains the latter because

$$\{i : i = O(\nu)\} \supseteq \{i : i = o(\nu)\}$$

and that both sets are symmetric by definition of the Landau notations, and through Lemma 1.28 item (4) extended to the nonstandard setting.

Lemma 4.2. *Let $a \in \mu$ and $\nu > \mathbb{N}$. The following conditions are equivalent:*

(1) a^i is defined and $a^i \in \mu$ for all $i \in \{1, \dots, \nu\}$
(2) $a \in G^o(\nu)$

(3) there is $\tau \in \{1, \ldots, \nu\}$ such that $\nu = O(\tau)$ and a^i is defined and $a^i \in \mu$ for all $i \in \{1, \ldots, \tau\}$

PROOF. (1) $\Rightarrow$ (2): Let σ be such that $\sigma = O(\nu)$, i.e. there is $n > 0$ satisfying $\sigma < n\nu$. We want to show that a^σ is defined and that $a^\sigma \in \mu$. We prove the first assertion by internal induction on i. We first prove that a^i is defined for all $i \in \{1, \ldots, \sigma\}$. This is true for all $i \leq \nu$, so now we suppose that this is true for some i such that $\nu < i + 1 \leq \sigma$. By Lemma 1.28, in order to show that a^{i+1} is defined, it suffices to show that $(a^k, a^l) \in \Omega^*$ for all $k, l \in \{1, \ldots, i\}$ with $k + l = i + 1$. However, $(a^k, a^l) \in \mu \times \mu \subseteq \Omega^*$, because μ is a group ; which finishes the induction.
We next show that if a^σ is defined, then $a^\sigma \in \mu$: we write $\sigma = m\nu + \eta$ for some m and $\eta < \nu$. By Lemma 1.28,

$$a^\sigma = (a^\nu)^m . a^\eta \in \mu.\mu \subseteq \mu$$

(2) $\Rightarrow$ (1) is clear.
(2) $\Rightarrow$ (3): Recall that ν is infinite (in the sense that $\nu > n$ for all $n \in \mathbb{N}$), and that there is no smallest infinite number : $\nu \neq 0$, so there is $\tau \in \mathbb{N}^*$ such that $\nu = \tau + 1$, as the corresponding statement is true in $\mathbb{N}$. This number τ cannot be finite, otherwise ν would be finite. It also verifies $\nu = O(\tau)$. The rest is obvious.
(3) $\Rightarrow$ (2): Suppose there is $\tau \in \{1, \ldots, \nu\}$ as in (3). By the proof of (1) $\Rightarrow$ (2), we see that $a \in G^o(\tau)$. Now as $i = O(\nu) \Rightarrow i = O(\tau)$, we have that $\{i : i = O(\tau)\} \supseteq \{i : i = O(\nu)\}$, so $G^o(\tau) \subseteq G^o(\nu)$, which concludes the proof. $\square$

DEFINITION 4.3. *An element $a \in \mu$ is said to be **degenerate** if, for all i, a^i is defined and $a^i \in \mu$.*

Note that this term is the one used in [**5**], but in [**20**] the term **parameter** is used for nondegenerate elements.

PROPOSITION 4.4. [**20**] *G is NSS iff μ has no degenerate elements other than* 1.

PROOF. G has no small subgroups iff G has no small cyclic subgroups.Let $j \in I$ such that O_j is a neighborhood of 1 which does not contain a nontrivial subgroup of G; then if $x_i \in O_j$ and $x_i \neq 1$, there is $n \in \mathbb{N}$ such that $x_i^n \notin O_j$ (i.e. $\neg P_j(x_i^n)$); otherwise x_i generates a group wholly contained in O_j. Considering the set $A_{x_i} = \{x_i^1, x_i^2, \ldots, x_i^n, \ldots\}$, this can also be written $A_{x_i} \nsubseteq O_j$. Let $x = \langle x_i \rangle \in \mu$. Then $x \in O_j^*$, and if $x \neq 1$, by transfer $\langle A_{x_i} \rangle \nsubseteq O_j^*$; i.e. there is $\nu \in \mathbb{N}^*$ with $x^\nu \notin O_j^*$. But then $x^\nu \notin \mu$, so x is not degenerate.

Conversely, suppose that every neighborhood O_i of 1 contains a nontrivial subgroup of G (which can be supposed to be of the form A_{x_i} as above). By κ-saturation there is an element $y \in \mu \setminus \{1\}$, such that for all $\nu \in \mathbb{N}^*$, $y^\nu \in \mu$, hence y is degenerate. $\square$

From now on, let U denote a compact symmetric neighborhood of 1 such that $U \subseteq U_2$.
(It is assumed that a corresponding symbol of predicate is added to the language.)

Definition 4.5 (**order**). *Let $a \in G^*$. If, for all i, a^i is defined and $a^i \in U^*$, define $ord_U(a) = \infty$. Else, define $ord_U(a) = \nu$ if, for all i with $|i| \leq \nu$, a^i is defined, $a^i \in U^*$, and ν is the largest element of $\mathbb{N}^*$ for which this happens.*

Lemma 4.6. *(1) $ord_U(a) = 0$ iff $a \notin U^*$.*
(2) $ord_U(a) > \mathbb{N}$ if $a \in \mu$.
(3) If $a \in \mu$ and $ord_U(a) = \nu \in \mathbb{N}^*$, then $a^{\nu+1}$ is defined.

Proof. (1) Immediate
(2) By Lemma 3.22
(3) Let $i, j \in \{1, \ldots, \nu\}$ such that $i + j = \nu + 1$. Then

$$(a^i, a^j) \in U^* \times U^* \subseteq U_2^* \times U_2^* \subseteq \Omega^*$$

Hence by Lemma 1.28, $a^{\nu+1}$ is defined.

□

Notice that $ord_U(a) = \nu$ because $a^{\nu+1}$ is not in U^*, not because $a^{\nu+1}$ is undefined.

Definition 4.7. *We say that $a \in \mu$ is U-**pure** if it is nondegenerate and $a \in G(\tau)$, where $\tau := ord_U(a)$. We say that $a \in \mu$ is **pure** if it is V-pure for some compact symmetric neighborhood V of 1 such that $V \subseteq U_2$.*

If U contains no nontrivial connected subgroup of G, then every $a \in \mu$ which is nondegenerate is U-pure (and thus if G is NSS, every $a \in \mu \setminus \{1\}$ is U-pure for all U):

Lemma 4.8. *Suppose $a \in \mu$ and $a^i \notin \mu$ for some $i = o(ord_U(a))$. Then U contains a nontrivial connected subgroup of G.*

Proof. By Lemma 3.23, the set

$$G_U(a) := \{st(a^i) : |i| = o(ord_U(a))\}$$

is a union of connected subsets of U, each containing 1, hence is connected. It is also a subgroup of G by Lemma 1.28. □

Lemma 4.9. *Let $a \in \mu$. Then a is pure iff there is $\nu > \mathbb{N}$ such that a^ν is defined, $a^\nu \notin \mu$ and $a \in G(\nu)$.*

Proof. First suppose that a is V-pure. Let $\nu = ord_V(a)$. Then by definition of the order a^ν is defined. Now since $a^{\nu+1}$ is defined (see Lemma 4.6), $a^\nu \notin \mu$, else $a^{\nu+1} = a^\nu.a$ would be in μ, contradicting the fact that $a^{\nu+1} \notin V^*$. However, $a^i \in \mu$ for $i = o(\nu)$ by definition of V-purity. Conversely, suppose one has $\nu > \mathbb{N}$ such that a^ν is defined, $a^\nu \notin \mu$ and $a \in G(\nu)$. Since $a^\nu \notin \mu$, there is a compact symmetric neighborhood V of 1 such that

$V \subseteq U_2$ and such that $a^\nu \notin V^*$. Hence $max\{k \in \mathbb{N}^* : a^k \in V^*\}$ exists, and we set $\tau := ord_V(a)$. Then $\tau < \nu$, implying that $a \in G(\tau)$ and thus a is V-pure. □

2. Special neighborhoods

In this subsection, it is assumed that G is NSS. Recall Lemma 1.25 which states the existence of open symmetric neighborhoods U_n of 1 for $n > 0$ such that $U_{n+1} \subseteq U_n$ and for all $(a_1, \dots, a_n) \in U_n^{\times n}$, $a_1 \dots a_n$ is defined.

Definition 4.10. *A* ***special neighborhood*** *of G is a compact symmetric neighborhood U of 1 in G such that $U \subseteq U_2$, U contains no nontrivial subgroup of G, and for all $x, y \in U$, if $x^2 = y^2$, then $x = y$.*

We will now show that G has a special neighborhood. We first look for an "almost locally invariant" neighborhood of 1, i.e. which is almost invariant by conjugation, for elements sufficiently close to identity. We begin by some Lemmas about local groups, which we have adapted from results on topological groups in [**13**].

Lemma 4.11. *Let F, $C \subseteq G$ such that F is closed, C is compact, $C \subseteq U_3$, and $F \cap C = \emptyset$. Then there is a neighborhood V of 1 such that $V \subseteq U_3$, and $F \cap CV = \emptyset$.*
Similarly, there is $V' \subseteq U_3$ such that $F \cap V'C = \emptyset$.

Proof. Let $x \in C$, x is thus in the open set $U_3 \setminus F \subseteq G \setminus F$. By the remark after Lemma 1.25, there is a neighborhood W_x of 1 such that $W_x \subseteq U_3$ and $W_x^2 \subseteq x^{-1}(U_3 \setminus F)$. By compacity of C, there is a set of points x_i, $i = 1, \dots, n$, and a set of associated neighborhoods W_{x_i} such that $x_i(W_{x_i})^2 \subseteq (U_2 \setminus F)$ and

$$\bigcup_{i=1}^{n} x_i W_{x_i} \supseteq C$$

Set

$$V := \bigcap_{i=1}^{n} W_{x_i}$$

Then $V \subseteq U_3$ and for any $x \in C$, there is some i, $1 \leq i \leq n$ such that $x \in x_i W_{x_i}$, so $xV \subseteq x_i W_{x_i}^2 \subseteq G \setminus F$. Hence $xV \cap F = \emptyset$ and thus $CV \cap F = \emptyset$.

□

Lemma 4.12. *Assuming that G is locally compact, let $K \subseteq U_{12}$ be a compact subset of G and let $O \subseteq U_3$ be an open neighborhood of 1. There exists a neighborhood $V \subseteq U_{12}$ of 1 in G such that for all $x \in K$, $xVx^{-1} \subseteq O$.*

Proof. Let $x \in K$. By Proposition 1.2 and local compacity of G, there is a compact neighborhood V_x of 1 such that $V_x \subseteq xOx^{-1} \cap U_{12}$. Then, setting $F := G \setminus O$, we obtain that $x^{-1}V_x x \cap F = \emptyset$. Consequently, as F is closed, by Lemma 4.11 and local compacity there is a compact symmetric neighborhood $W_x^0 \subseteq U_{12}$ of 1 such that $x^{-1}V_x x W_x^0 \cap F = \emptyset$. Since the product of two compact sets is compact, and since the image by a continuous application of a compact is compact,we get that $x^{-1}V_x x W_x^0$ is compact, so we can apply a second time Lemma 4.11, hence there is a compact symmetric neighborhood $W_x^1 \subseteq U_5$ of 1 such that

$$W_x^1 x^{-1} V_x x W_x^0 \cap F = \emptyset$$

Let $W_x := W_x^0 \cap W_x^1$. Then $W_x x^{-1} V_x x W_x \cap F = \emptyset$. For any $y \in xW_x$, $y^{-1} \in W_x x^{-1}$ and $y^{-1} V_x y \subseteq O$.

By compacity of K, there is some finite subset X of G such that

$$K \subseteq \bigcup_{x \in X} xW_x$$

where each W_x is associated with V_x, as constructed above.

We set $V := \bigcap_{x \in X} V_x$. Consequently, if $y \in K$, then $y \in xW_x$ for some $x \in X$ and

$$y^{-1} V y \subseteq W_x x^{-1} V_x x W_x \subseteq O$$

which ends the proof. $\square$

LEMMA 4.13. *Suppose that G is NSS and locally compact. There is a neighborhood V of 1 such that $V \subseteq U_2$ and for all $x, y \in V$, if $x^2 = y^2$, then $x = y$.*

PROOF. Let O be a compact symmetric neighborhood of 1 in G such that O contains no non trivial subgroup of G and $O \subseteq U_{12}$. We choose a symmetric open neighborhood $W \subseteq U_3$ of 1 in G such that $W^5 \subseteq O$.
By Lemma 4.12 and since O is compact, there is an open symmetric neighborhood $V \subseteq U_{12}$ of 1 such that $V^2 \subseteq W$ and for all $g \in O$, $gVg^{-1} \subseteq W$.
We next show that this V fulfills the desired condition. Let $x, y \in V$ such that $x^2 = y^2$, and let $a := x^{-1}y \in V^2 \subseteq W \subseteq O$. To show that $a = 1$, we will show that $a^{\mathbb{Z}} \subseteq V$, which implies that $a^{\mathbb{Z}}$ is trivial since O contains no nontrivial subgroup of G. As $V \subseteq U_5$, we have that

$$axa = (x^{-1}y)x(x^{-1}y) = x^{-1}y^2 = x^{-1}x^2 = x$$

Next step is to show that for all n, a^n is defined, $a^n \in O$ and $a^n = xa^{-n}x^{-1}$, which we will do by induction. We have already shown the case $n = 1$. Suppose the assertion is true for all $m \in \{1, \ldots, n\}$. To show that a^{n+1} is defined, we can show that $(a^i, a^j) \in \Omega$ for all $i, j \in \{1, \ldots, n\}$ such that $i + j = n + 1$ thanks to Lemma 1.28. Indeed, by the inductive hypothesis, $(a^i, a^j) \in O \times O \subseteq \Omega$, which proves that a^{n+1} is defined. We then have, as $O \subseteq U_6$:

$$a^{n+1} = a^n.a = (xa^{-n}x^{-1}).(xa^{-1}x^{-1}) = xa^{-n-1}x^{-1}$$

The last equality following Lemma 1.28 item (4). Now we want to show that $a^{n+1} \in O$. There are two cases. The first one is when $n+1$ is even, say $n+1 = 2m$. Then as $x \in V$ which is symmetric, $x^{-1} \in V$; moreover $a^m \in O$

by induction hypothesis, so a^{-m} is also in O which is symmetric. Since for all $g \in O$, $gVg^{-1} \subseteq W$, we get that $a^{-m}x^{-1}a^m \in W$, and finally:

$$a^{n+1} = a^m.a^m = xa^{-m}x^{-1}a^m \in xW \subseteq W^2 \subseteq O$$

If $n+1$ is odd, then $a^{n+1} = a^n.a \in W^2.W^2 \subseteq O$.

This yields a subgroup $a^{\mathbb{Z}}$ of G, with $a^{\mathbb{Z}} \subseteq O$. Thus $a = 1$ and so $x = y$. □

Until further notice, **let us fix $\mathcal{U}$ a special neighborhood of 1 in** G, as we know its existence by the last Lemma. Notice that then every $a \in \mu \setminus \{1\}$ is $\mathcal{U}$-pure.

$$\exists \nu \in \mathbb{N}^*(a^\nu \notin \mu)$$

For $a \in G^*$, we set $ord(a) := ord_{\mathcal{U}}(a)$, the latter being the biggest nonstandard integer ν such that $a^\nu \in \mathcal{U}^*$.

Lemma 4.14. *Let $a \in G^*$. Then $ord(a)$ is infinite iff $a \in \mu$.*

Proof. If $a \in \mu$, then $ord(a)$ is infinite thanks to Lemma 3.22. Conversely, suppose $ord(a)$ is infinite. Then, by definition of $ord_{\mathcal{U}}(a)$, we get that for all $k \in \mathbb{Z}$, a^k is defined and $a^k \in \mathcal{U}^*$. As $\mathcal{U}$ is compact, $a^k \in G^*_{ns}$ (Theorem 3.15), so in particular $st(a^k)$ is defined. We see by induction and with the help of Lemma 3.21 that $st(a^k) = (st(a))^k$, which is then in $\mathcal{U}$ for all $k \in \mathbb{Z}$. Since $\mathcal{U}$ is a special neighborhood, this implies that $st(a) = 1$, i.e. $a \in \mu$. □

3. A countable neighborhood basis of the identity

Suppose G to be NSS.

Let Q be an internal subset of G^* such that $Q \subseteq \mu$, $1 \in Q$ and Q is symmetric. If ν is such that for all internal sequence $a_1, \dots, a_\nu$ of elements of Q, $a_1.\dots.a_\nu$ is defined, we set Q^ν to be the internal subset of G^* containing the products $a_1.\dots.a_\nu$. In this case we say that Q^ν is defined.
If, for all ν, Q^ν is defined and $Q^\nu \subseteq \mu$, we say that Q is degenerate. Let $U \subseteq U_2$ be a compact symmetric neighborhood of 1. If, for all ν, Q^ν is defined and $Q^\nu \subseteq U^*$, we define $ord_U(Q) = \infty$. Otherwise, we define $ord_U(Q) = \nu$ if Q^ν is defined, $Q^\nu \subseteq U^*$, and ν is the biggest element of $\mathbb{N}^*$ for which this happens.
Like in Lemma 3.22, $ord_U(Q) > \mathbb{N}$ and if $ord_U(Q) \in \mathbb{N}^*$, then $Q^{ord_U(Q)+1}$ is defined.

We now set $ord(Q) = ord_{\mathcal{U}}(Q)$, $\mathcal{U}$ a special neighborhood.
We extend it to the standard case in the following way: for a symmetric set $P \subseteq G$ such that $1 \in G$, we define $ord(P)$ as the biggest integer n such that P^n is defined and $P^n \subseteq \mathcal{U}$. If there is not such a n, we set $ord(P) = \infty$. Set

$$V_n := \{x \in G : ord(x) \geq n\}$$

Recall that $p_n : G \to G$, $a \mapsto a^n$ if a^n is defined, is a continuous map with open domain containing U_n.

LEMMA 4.15. *(1) For all n, $p_1^{-1}(\mathcal{U}) \cap \cdots \cap p_n^{-1}(\mathcal{U}) \subseteq V_n$*

(2) $(V_n : n \geq 1)$ is a decreasing sequence of compact symmetric neighborhoods of 1 in G

(3) $ord(V_n) \to \infty$ as $n \to \infty$

(4) $\{V_n : n \geq 1\}$ is a countable neighborhood basis of 1 in G

PROOF. If $x \in p_1^{-1}(\mathcal{U}) \cap \cdots \cap p_n^{-1}(\mathcal{U})$, $ord(x) \geq n$ so $x \in V_n$. To check the other items, let $\sigma \geq \mathbb{N}$ and consider the set

$$V_\sigma := \{g \in G^* : ord(g) \geq \sigma\}$$

Notice that V_σ is well an internal set, as it can be written $V_\sigma = \langle V_{\sigma_n} \rangle$, where $\langle \sigma_n \rangle =: \sigma$. As we are still under the hypothesis that G is NSS, Lemma 4.14 brings that $V_\sigma \subseteq \mu$. Thus, for all m and all $x_1, \ldots, x_m \in V_\sigma$, the product $x_1 \ldots x_m$ belongs to μ which is a group. In particular, if U is a neighborhood of 1 in G, $x_1 \ldots x_m \in U^*$; i.e. $(V_\sigma)^m \subseteq U^*$. Hence by transfer, if we take n sufficiently big, $(V_n)^m$ is defined for all m and is included in the neighborhood U, which shows item (3).

□

CHAPTER 5

Local 1-parameter subgroups

Standard local 1-parameter subgroups are first defined, and the notion is adapted to the nonstandard setting. Then local 1-parameter subgroups are constructed from pure infinitesimals, using previous results. At the end the local exponential map is introduced. This chapter is based on [**5**]; proofs are more detailed.

1. Standard local 1-parameter subgroups

DEFINITION 5.1. *A **local** 1**-parameter subgroup** of G, abbreviated **local** 1**-ps** of G, is a continuous map $X : (-r, r) \to G$, for some $r \in (0, \infty)$, such that*

(1) image $(X) \subseteq \Lambda$, and
(2) if $r_1, r_2, r_1 + r_2 \in (-r, r)$, then $(X(r_1), X(r_2)) \in \Omega$ and

$$X(r_1 + r_2) = X(r_1).X(r_2)$$

Hence if $X : (-r, r) \to G$ is a local 1-ps of G, $s \in (-r, r)$ and n an integer such that $ns \in (-r, r)$, then $X(s)^n$ is defined and $X(ns) = X(s)^n$.

DEFINITION 5.2. *Let X, Y be local 1-parameter subgroups of G. We say that X is equivalent to Y if there is $r \in \mathbb{R}^{>0}$ such that*

$$r \in \textit{domain}(X) \cap \textit{domain}(Y) \textit{ and } X_{|(-r,r)} = Y_{|(-r,r)}$$

We let $[X]$ denote the equivalence class of X with respect to this equivalence relation. We also let

$$L(G) := \{[X] : X \textit{ is a local 1-ps of } G\}$$

In other words, $L(G)$ is the set of germs at 0 of local 1-parameter subgroups of G. It is possible to define a 'scalar' multiplication:

$$\begin{array}{ccc} \mathbb{R} \times L(G) & \to & L(G) \\ (s, [X]) & \mapsto & s.[X] \end{array}$$

as follows: let $X \in [X]$ with $X : (-r, r) \to G$. If $s = 0$, then $0.X = O$, where

$$O : \begin{array}{ccc} \mathbb{R} & \to & G \\ t & \mapsto & 1 \end{array}$$

If $s \neq 0$, we define

$$sX : \begin{array}{ccc} (\frac{-r}{|s|}, \frac{r}{|s|}) & \to & G \\ t & \mapsto & X(st) \end{array}$$

Then sX is a local 1-ps of G, and we let $s.[X] := [sX]$. It is obvious that the definition is independant from the choice of a representative. We see that for all $[X] \in L(G)$ and $s, s^{'} \in \mathbb{R}$, $1.[X] = [X]$ and $s.(s^{'}.[X]) = (ss^{'}).[X]$.

Let X_1 and X_2 be local 1-parameter subgroups of G such that $[X_1] = [X_2]$, and let $t \in \text{domain}(X_1) \cap \text{domain}(X_2)$. As X_1 and X_2 are equivalent, there exists a common restriction of their domains on which they are equal: so let $r \in \mathbb{R}^{>0}$ be such that $X_{1|(-r,r)} = X_{2|(-r,r)}$ and choose $n > 0$ such that $\frac{t}{n} \in (-r, r)$. Then

$$X_1(t) = (X_1(\frac{t}{n}))^n = (X_2(\frac{t}{n}))^n = X_2(t)$$

Whence $X_1(t) = X_2(t)$.

Henceforth, as we have the equality on the intersection of two different domains, it makes sense to define, for $[X] \in L(G)$,

$$\text{domain}([X]) := \bigcup_{X \in [X]} \text{domain}(X)$$

2. From pure infinitesimals to local 1-parameter subgroups

Let U be a compact symmetric neighborhood of 1 such that $U \subseteq U_2$, as defined in Section 1 from Chapter 4, and let $a \in \mu$ being U-pure, $\tau := ord_U(a)$ and $0 < \nu = O(\tau)$. Let $r \in \mathbb{R}^{>0}$. Then $r\nu$ can be written as the equivalence class $\langle (r\nu)_i \rangle$. We can thus define the nonstandard integer part $floor(r\nu)$ componentwise :

$$floor(r\nu) = \lfloor r\nu \rfloor := \langle \lfloor (r\nu)_i \rfloor \rangle \text{, where } \lfloor (r\nu)_i \rfloor = max\{n \in \mathbb{N} : n \leq (r\nu)_i\}$$

We define the set :

$$\Sigma_{\nu,a,U} := \{r \in \mathbb{R}^{>0} : a^{\lfloor r\nu \rfloor} \, \text{is defined and} \, a^i \in U^* \, \text{if} \, |i| \leq \lfloor r\nu \rfloor\}$$

Since $\nu = O(\tau)$, there is, by definition, $m \in \mathbb{N} \setminus \{0\}$ such that $\nu < m\tau$. Let $r_o := 1/m$. In this case $\lfloor r_0 \nu \rfloor \leq \tau = ord_U(a)$, and thus, as a is U-pure, it shows that $\Sigma_{\nu,a,U} \neq \emptyset$.

Now let $r_{\nu,a,U} := sup \Sigma_{\nu,a,U}$, with the convention that $r_{\nu,a,U} = \infty$ if $\Sigma_{\nu,a,U} = \mathbb{R}^{>0}$. Suppose $s < r_{\nu,a,U}$. Then $s \in \Sigma_{\nu,a,U}$ and $a^{\lfloor s\nu \rfloor} \in U^*$, and $a^{\lfloor s\nu \rfloor + 1}$ is defined: let i, j be such that $i + j = \lfloor s\nu \rfloor + 1$. By construction of the set $\Sigma_{\nu,a,U}$, a^i, a^j are then in U^*, hence

$$(a^i, a^j) \in U^* \times U^* \subseteq U_2^* \times U_2^* \subseteq \Omega^*$$

so $a^{i+j} = a^{\lfloor s\nu \rfloor + 1}$ is defined. Then $a^{\lfloor s\nu \rfloor + 1} = a^{\lfloor s\nu \rfloor}.a \in U^*.\mu \subseteq G^*_{ns}$ (Recall that $U^* \subseteq G^*_{ns}$ because U is compact; see Theorem 3.15).
By definition of the floor function, for $s \in (-r_{\nu,a,U}, 0)$, we have that

$$\lfloor s\nu \rfloor = -\lfloor (-s)\nu \rfloor \text{ or } -\lfloor (-s)\nu \rfloor - 1$$

Hence for such s, we have that $a^{\lfloor s\nu \rfloor}$ is defined and nearstandard as well.

Then observe that when both defined, $st(a^{\lfloor s\nu \rfloor}) = st(a^{\lfloor s\nu \rfloor+1})$:
$a^{\lfloor s\nu \rfloor+1} = a^{\lfloor s\nu \rfloor}.a$ hence, as $a \in \mu$,

$$st(a^{\lfloor s\nu \rfloor+1}) = st(a^{\lfloor s\nu \rfloor}).st(a) = st(a^{\lfloor s\nu \rfloor})$$

Similarly, $st(a^{\lfloor s\nu \rfloor-1}) = st(a^{\lfloor s\nu \rfloor})$.

We are now ready for the following result :

LEMMA 5.3. *The map* $X : (-r_{\nu,a,U}, r_{\nu,a,U}) \to G$ *given by* $X(s) := st(a^{\lfloor s\nu \rfloor})$ *is a local* 1*-ps of* G.

PROOF. Let $r := r_{\nu,a,U}$. The preceding arguments have already shown that image$(X) \subseteq U_2$. Let s_1, s_2 be such that $s_1, s_2, s_1+s_2 \in (0,r)$. First note that

$$\forall x, y \in \mathbb{R}, \lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$$

thus

$$\lfloor s_1\nu \rfloor + \lfloor s_2\nu \rfloor = \lfloor (s_1+s_2)\nu \rfloor \text{ or } \lfloor (s_1+s_2)\nu \rfloor - 1$$

By construction of $\Sigma_{\nu,a,U}$, $a^{\lfloor s_1\nu \rfloor}, a^{\lfloor s_2\nu \rfloor}$ and $a^{\lfloor (s_1+s_2)\nu \rfloor}$ are defined and belong to U^*. As

$$a^{\lfloor s_1\nu \rfloor + \lfloor s_2\nu \rfloor} = a^{\lfloor (s_1+s_2)\nu \rfloor} \text{ or } a^{\lfloor (s_1+s_2)\nu \rfloor - 1}$$

we obtain that

$$\begin{aligned} X(s_1+s_2) &= st(a^{\lfloor (s_1+s_2)\nu \rfloor}) \\ &= st(a^{\lfloor s_1\nu \rfloor + \lfloor s_2\nu \rfloor}) \\ &= st(a^{\lfloor s_1\nu \rfloor}.a^{\lfloor s_2\nu \rfloor}) \\ &= st(a^{\lfloor s_1\nu \rfloor}).st(a^{\lfloor s_2\nu \rfloor}) = X(s_1).X(s_2) \end{aligned}$$

Now we consider the case $s_1, s_2, s_1+s_2 \in (-r,r)$ with $s_1.s_2 < 0$. We use the last item of Lemma 1.28:
by definition, $a^{\lfloor s_1\nu \rfloor}, a^{\lfloor s_2\nu \rfloor}$ are defined and in U^*, thus

$$st(a^{\lfloor s_1\nu \rfloor}), st(a^{\lfloor s_2\nu \rfloor}) \in U \subseteq U_2$$

i.e. $(st(a^{\lfloor s_1\nu \rfloor}), st(a^{\lfloor s_2\nu \rfloor})) \in \Omega$ and we can apply the Lemma, which gives us

$$st(a^{\lfloor (s_1+s_2)\nu \rfloor}) = st(a^{\lfloor s_1\nu \rfloor}).st(a^{\lfloor s_2\nu \rfloor})$$

i.e. $X(s_1+s_2) = X(s_1).X(s_2)$.

It is easy to see that $X(-s) = X(s)^{-1}$, so for $s_1, s_2, s_1 + s_2 \in (-r, 0)$, we have

$$\begin{aligned} X(s_2 + s_1) &= X(-(-s_1 - s_2)) \\ &= X(-s_1 - s_2)^{-1} \\ &= (X(-s_1).X(-s_2))^{-1} \\ &= (X(s_1)^{-1}.X(s_2)^{-1})^{-1} = X(s_2).X(s_1) \end{aligned}$$

To show that X is continuous, we show that it is continuous at 0, which is sufficient by Lemma 1.14. Let V be a neighborhood of 1 in G. As G is supposed to be locally compact, there is $V_0 \subseteq V$ which is a compact neighborhood of 1. Let $i = o(\nu)$, i.e. $\forall n \in \mathbb{N}, |i| < \frac{\nu}{n}$. Then $\forall n \in \mathbb{N}, |\frac{i}{\nu}| < \frac{1}{n}$. If n is such that $1/n < r$, $|i| \leq \lfloor r\nu \rfloor$ and a^i is defined and $a^i \in U^*$. Furthermore, if $i = o(\nu)$, then $i = o(\tau)$ because $\nu = O(\tau)$. Hence, as $a \in G(\tau)$, we get that $a^i \in \mu \subseteq V_0^*$. Therefore $X(s) \in V$ for s such that $|s| < \frac{1}{n}$; i.e. X is continuous. □

From now on, let $I := [-1, 1] \subseteq \mathbb{R}$.

LEMMA 5.4. *Suppose $X : I \to G$ is a continuous function such that for all $r, s \in I$, if $r + s \in I$, then $(X(r), X(s)) \in \Omega$ and $X(r + s) = X(r).X(s)$. Further suppose that $X(I) \subseteq U_4$. Then there exists $\epsilon \in \mathbb{R}^{>0}$ and a local 1-ps $\overline{X} : (-1 - \epsilon, 1 + \epsilon) \to G$ of G such that $\overline{X}|I = X$.*

PROOF. Omitted. One can see [**5**] for a proof of this Lemma ; the proof is not difficult but quite long. □

3. The set $L(G)$

Let G be NSS, and let $\mathcal{U}$ be a special neighborhood. We first set a kind of recapitulative Lemma.

LEMMA 5.5. *Suppose $\sigma > \mathbb{N}$ and $a \in G(\sigma)$. Then:*

1. *If $a \neq 1$, then a is $\mathcal{U}$-pure and $\sigma = O(ord(a))$;*
2. *If $i = o(\sigma)$, then a^i is defined and $a^i \in \mu$;*
3. *Let $\Sigma_a := \Sigma_{\sigma,a,\mathcal{U}}$ and $r_a := r_{\sigma,a,\mathcal{U}}$. Let $X_a : (-r_a, r_a) \to G$ be defined by $X_a(s) := st(a^{\lfloor s\sigma \rfloor})$. Then X_a is a local 1-ps of G.*

PROOF.
(1) If $ord(a) = o(\sigma)$, then $a^{ord(a)} \in \mu$, and, since μ is a subgroup, $a^{ord(a)+1} \in \mu \subseteq \mathcal{U}$, which is a contradiction. Hence $\sigma = O(ord(a))$.
(2) By the above item, if $i = o(\sigma)$, then $i = o(ord(a))$, whence a^i is defined and $a^i \in \mu$ since a is $\mathcal{U}$-pure.
(3) Follows from Lemma 5.3

□

LEMMA 5.6. *Suppose G is not discrete. Then $L(G) \neq \{\mathbb{O}\}$.*

Recall that $\mathbb{O} = [O]$, where $O : \mathbb{R} \to G$, $t \mapsto 1$.

PROOF. Let $a \in \mu \setminus \{1\}$, and $\sigma := ord(a)$. Because $a \in \mu$, we must have $\sigma > \mathbb{N}$; because $a \in G(\sigma)$, we must have $[X_a] \in L(G)$, where X_a is as defined in Lemma 5.5. Our aim is to show that $[X_a] \neq \mathbb{O}$. We will do it by showing that for every $n \in \mathbb{N}$ such that $\frac{1}{n} \in \Sigma_a$, $a^{\left\lfloor \frac{1}{n}\sigma \right\rfloor} \notin \mu$:

Let $t := \left\lfloor \frac{1}{n}\sigma \right\rfloor$. Then $t = \frac{1}{n}\sigma - \epsilon$, with $\epsilon \in [0,1)^*$. Towards a contradiction, suppose that $a^t \in \mu$. Then since $nt \leq \sigma$, a^{nt} is defined and $a^{nt} = (a^t)^n \in \mu$ by Lemma 1.28. Also $n\epsilon = \sigma - nt \in \mathbb{N}^*$ and $n\epsilon < n$, whence $a^{n\epsilon} \in \mu$. But then $a^\sigma = a^{nt+n\epsilon} = a^{nt}.a^{n\epsilon} \in \mu$, a contradiction (otherwise $a^{ord(a)+1} \in \mu \subseteq \mathcal{U}$). ☐

LEMMA 5.7. *Let $a \in G(\sigma) \setminus \{1\}$. Then:*

(1) $a^{-1} \in G(\sigma)$ and $[X_{a^{-1}}] = (-1).[X_a]$
(2) $b \in \mu \Rightarrow bab^{-1} \in G(\sigma)$ and $[X_{bab^{-1}}] = [X_a]$
(3) $[X_a] = \mathbb{O} \Leftrightarrow a \in G^o(\sigma)$
(4) $L(G) = \{[X_b] \,|b \in G(\sigma)\}$

PROOF. (1) μ is a subgroup so if $a^i \in \mu$, we have also $(a^i)^{-1} = (a^{-1})^i \in \mu$.

$$\begin{aligned} -X_a(t) &= X_a(-t) \\ &= st(a^{\lfloor -\sigma t \rfloor}) \\ &= st(a^{-\lfloor \sigma t \rfloor}) = X_{a^{-1}}(t) \end{aligned}$$

(2) Let $b \in \mu, \tau = ord(a)$ and $a \in G(\sigma) \setminus \{1\}$. Then, by Lemma 5.5 item (1), we get $\sigma = O(ord(a))$. Set $i = o(ord(a))$. We know that $a^i \in \mu$ and that $ba^ib^{-1} \in \mu$ because μ is a group. Suppose $ba^ib^{-1} = (bab^{-1})^i$. Therefore

$$\begin{aligned} (bab^{-1})^{i+1} &= (bab^{-1})^i.bab^{-1} \\ &= ba^ib^{-1}.bab^{-1} = ba^{i+1}b^{-1} \end{aligned}$$

Hence $(bab^{-1})^i$ is in μ, thus $bab^{-1} \in G(\sigma)$. If $r \in \Sigma_a$, one also have that $(bab^{-1})^{\lfloor r\sigma \rfloor}$ is defined and equals $ba^{\lfloor r\sigma \rfloor}b^{-1}$, whence

$$\begin{aligned} st((bab^{-1})^{\lfloor r\sigma \rfloor}) &= st(ba^{\lfloor r\sigma \rfloor}b^{-1}) \\ &= st(b)st(a^{\lfloor r\sigma \rfloor})st(b^{-1}) = st(a^{\lfloor r\sigma \rfloor}) \end{aligned}$$

because $st(b) = st(b^{-1}) = 1$ as the two elements b, b^{-1} are in μ. Finally $[X_{bab^{-1}}] = [X_a]$.

(3) We use Lemmas 4.2 and 5.5 item (1). Recall that $a \in G^o(\sigma)$ iff, for $r = O(ord(a))$, a^r is defined and $a^r \in \mu$. By Lemma 5.5, $\sigma = O(ord(a))$. Let $s \in (-r_a, r_a)$. Then $a^{\lfloor s\sigma \rfloor}$ is defined and

$\lfloor s\sigma \rfloor = O(ord(a))$. Hence $a^{\lfloor s\sigma \rfloor} \in \mu$, and then $X_a = st(a^{\lfloor s\sigma \rfloor}) = 1$. If $X_a(s) = 1$, $a^{\lfloor s\sigma \rfloor} \in \mu$, and by Lemma 4.2, $a \in G^o(\sigma)$.

(4) Suppose $\mathbb{X} \in L(G)$ and $X \in \mathbb{X}$, and let $(-r, r) := domain(X)$. We then consider the nonstandard extension of X:

$$\overline{X} : \begin{array}{ccc} (-r,r)^* & \to & G^* \\ \langle s_i \rangle & \mapsto & \left\langle st(a^{\lfloor s_i\sigma \rfloor}) \right\rangle \end{array}$$

$\sigma = \langle \sigma_i \rangle$, $\frac{1}{\sigma} \in (-1,1)^*$. Set $c := \overline{X}(\frac{1}{\sigma}) \in \mu$, $s < min\{r, r_c\}$ and let ϵ be an infinitesimal element of $\mathbb{R}^*$ (i.e. $\epsilon \in \mu(0)$) such that, now considering $s \in \mathbb{R}^*$, we can write $s = \frac{\lfloor s\sigma \rfloor}{\sigma} + \epsilon$.

$$\frac{\lfloor s\sigma \rfloor}{\sigma} \leq s < \frac{\lfloor s\sigma \rfloor}{\sigma} + \frac{1}{\sigma}$$

We will now show that $\mathbb{X} = [X_c]$.

$$\begin{aligned} X(s) &= st(\overline{X}(\frac{\lfloor s\sigma \rfloor}{\sigma} + \epsilon)) \\ &= st(\overline{X}(\frac{\lfloor s\sigma \rfloor}{\sigma})\overline{X}(\epsilon)) \\ &= st(\overline{X}(\frac{\lfloor s\sigma \rfloor}{\sigma})).st(\overline{X}(\epsilon)) \\ &= st(\overline{X}(\frac{1}{\sigma})^{\lfloor s\sigma \rfloor}).1 \\ &= st(c^{\lfloor s\sigma \rfloor}) \\ &= X_c(s) \end{aligned}$$

Hence X and X_c agree on $(-r,r) \cap (-r_c, r_c)$.

$\square$

4. The local exponential map

Let G be NSS and $\mathcal{U}$ be a special neighborhood. Recall that every $a \in \mu \setminus \{1\}$ is then $\mathcal{U}$-pure, that $I = [-1,1] \subseteq \mathbb{R}$, and that $L(G) = \{\mathbb{X} = [X] : X \text{ is a local } 1-ps \text{ of } G\}$;

$$domain\, \mathbb{X} = \bigcup_{X \in \mathbb{X}} domain(X)$$

If $domain(X) = (-r,r)$ then for $ns \in (-r,r)$, $X(ns) = X(s)^n$, and a scalar multiplication is defined by setting:
for $s \neq 0$, $sX : (\frac{-r}{|s|}, \frac{r}{|s|}) \to G$; $t \mapsto X(st)$,
for $s = 0$, $s.X = O$, where $O : \mathbb{R} \to G$; $t \mapsto 1$.

We consider the following sets:

$$\mathcal{K} := \{\mathbb{X} \in L(G) | I \subseteq domain(\mathbb{X}) \text{ and } \mathbb{X}(I) \subseteq \mathcal{U}\}$$

and

$$K := \{\mathbb{X}(1) | \mathbb{X} \in \mathcal{K}\}$$

We first show that $\mathcal{K}$ "homotethically" covers all of $L(G)$.

LEMMA 5.8. *For every $\mathbb{X} \in L(G)$, there is $s \in (0,1)$ such that $s.\mathbb{X} \in \mathcal{K}$.*

PROOF. (1) Let $\mathbb{X} \in L(G)$, and $X : (-r, r) \to G$ be a representative of $\mathbb{X}$.
- If $I \nsubseteq (-r, r)$, i.e. $r \leq 1$. We take s_1 such that $0 < s_1 < r \leq 1$, so
$$1 \leq \frac{1}{r} < \frac{1}{s_1} \text{ and } r \leq 1 < \frac{r}{s_1}$$
Then $s_1.X : (\frac{-r}{s_1}, \frac{r}{s_1}) \to G$ contains I in its domain.
- If $r > 1$, we take $s_1 = 1$.

(2) Now we want the image of I to be included in $\mathcal{U}$. Since the map $s_1.X$ is continuous, there is s_2 such that $0 < s_2 < 1$, $s_2 I$ is a neighborhood of 0 and $s_1.X(s_2 I) \subseteq \mathcal{U}$, which can also be written $(s_2.(s_1.X))(I) \subseteq \mathcal{U}$. At the end we set $s := s_1.s_2$, and we have $s.\mathbb{X} \in \mathcal{K}$.

□

LEMMA 5.9. *The map $\mathcal{K} \to K$, $\mathbb{X} \mapsto \mathbb{X}(1)$ is bijective.*

PROOF. The surjectivity is clear by construction of K. To check the injectivity, let $\mathbb{X}_1, \mathbb{X}_2 \in \mathcal{K}$, with representatives X_1, X_2, chosen in such a way that I is included in both their domains (it is possible by definition of *domain*$(\mathbb{X})$). Suppose $X_1(1) = X_2(1)$. Then $X_1(\frac{1}{2})^2 = X_2(\frac{1}{2})^2$ and so $X_1(\frac{1}{2}) = X_2(\frac{1}{2})$ since $\mathcal{U}$ is a special neighborhood. Inductively, we obtain $X_1(\frac{1}{2^n}) = X_2(\frac{1}{2^n})$ for all n, hence for $k \in \mathbb{Z}$ such that $\frac{k}{2^n} \in I$: $X_1(\frac{k}{2^n}) = X_2(\frac{k}{2^n})$. The family $(\frac{k}{2^n})$, as defined above, is dense in I around 0, so we can find restrictions of X_1 and X_2 which are equal because they are continuous. Finally $[X_1] = [X_2]$. □

From now on, we define the local exponential map to be:

$$\begin{aligned} E : &\mathcal{K} \to K \\ &\mathbb{X} \mapsto \mathbb{X}(1) \end{aligned}$$

CHAPTER 6

Local Gleason-Yamabe Lemmas

These Lemmas will be used later to put a group structure on $L(G)$. They show that multiplication of small elements is almost commutative and that given a set A of small elements, one cannot move away from the identity faster by using products of elements of A than by using powers of a single element of A. They are shown by looking at elements of G acting on the space of continuous functions from G to $\mathbb{R}$, the support of which is compact. In this chapter we will mainly follow the article [**5**], in which I.Goldbring give a local adaptation of the original Lemmas from Gleason and Yamabe; we give more detailed proofs. The section on the existence of a Haar measure is based on [**8**] and on [**13**].

Along this chapter, we let $\mathcal{U}$ and $\mathcal{W}$ be two neighborhoods of 1 in G,both symmetric and compact, and such that $\mathcal{U}^P \subseteq \mathcal{W} \subseteq U_M$ for $P, M \in \mathbb{N}$ two suitably chosen integers. We also let $Q \subseteq \mathcal{U}$ be a symmetric neighborhood of 1 such that $N := ord_{\mathcal{U}}(Q) \neq \infty$. Then Q^{N+1} is defined. We then set the map $\Delta := \Delta_Q$:

$$\Delta : \begin{array}{ccl} G & \to & [0,1] \\ 1 & \mapsto & 0 \\ x & \mapsto & \left\{ \begin{array}{ccll} \frac{i}{N+1} & \text{if} & x \in Q^i \setminus Q^{i-1} & \text{for } 1 \leq i \leq N \\ 1 & \text{if} & x \notin Q^N & \end{array} \right. \end{array}$$

Then, for all $x \in G$, we get:

- $\Delta(x) = 1$ if $x \notin \mathcal{U}$ (because in that case, $x \notin Q^N \subseteq \mathcal{U}$)
- For $a \in Q$ such that $(a, x) \in \Omega$, $|\Delta(ax) - \Delta(x)| \leq \frac{1}{N+1}$
  In fact, if $x \in Q^i, ax \in Q^{i+1}$ and $|\Delta(ax) - \Delta(x)| = \left|\frac{i+1-i}{N+1}\right|$.
  If $x = 1, a \neq 1$, $|\Delta(ax) - \Delta(x)| = \left|\frac{1}{N+1} - 0\right|$.
  If $x \in Q^N$ and $ax \in Q^{N+1}$, $|\Delta(ax) - \Delta(x)| = \left|\frac{N+1-N}{N+1}\right|$.
  If $x \notin Q^N$ and $ax \notin Q^N$, $|\Delta(ax) - \Delta(x)| = 0$.
  If $x = 1 = ax$, $|\Delta(ax) - \Delta(x)| = 0$.

Now we want to smooth out Δ. Recall that a topological space which is locally compact and T_2 is Tychonoff and consequently completely regular. This is the case of our local group G, so it is possible to find a continuous function $\tau : G \to [0, 1]$ satisfying $\tau(1) = 1$ and $\tau(x) = 0$ for all $x \in G \setminus \mathcal{U}$

(see definition 1.1).

Note that it will be important later that τ depends on $\mathcal{U}$ and not on Q.

Next set the map $\theta := \theta_Q : G \to [0,1]$ as follows :

$$\theta(x) = \begin{cases} sup\{(1-\Delta(y))\tau(y^{-1}x) : y \in \mathcal{U}\} & \text{if} \quad x \in \mathcal{W} \\ 0 & \text{if} \quad x \in G \setminus \mathcal{W} \end{cases}$$

LEMMA 6.1. *The following properties hold for the above defined functions.*

(1) $\theta(x) = 0$ *if* $x \notin \mathcal{U}^2$
(2) θ *is continuous*
(3) $0 \leq \tau \leq \theta \leq 1$
(4) $|\theta(ax) - \theta(x)| \leq \frac{1}{N}$ *for* $a \in Q$ *if* $(a, x) \in \Omega$

PROOF. (1) To show that $x \notin \mathcal{U}^2 \Rightarrow \theta(x) = 0$, suppose $\theta(x) \neq 0$. Hence there is $y \in \mathcal{U}$ such that

$$(1 - \Delta(y))\tau(y^{-1}x) \neq 0$$

$$\text{so } \tau(y^{-1}x) \neq 0 \text{ and } \Delta(y) \neq 1$$

then $y^{-1}x \in \mathcal{U}$ and $y \in Q^N \subseteq \mathcal{U}$, therefore $x = yy^{-1}x \in \mathcal{U}^2$

(2) To establish the continuity of θ, we consider its extension in G^*. Let $x \in G$. By Proposition 3.17, it suffices to show that $\theta(\mu(x)) \subseteq \mu(\theta(x))$. Recall that in G we have $\mu(x) = \mu x$. An element in $\mu(x)$ is then of the form ax, with $a \in \mu$. Similarly $\mu(\theta(x)) = \mu(0) + \theta(x)$ in $\mathbb{R}^*$, so we are looking for an element $b \in \mu(0)$ in $\mathbb{R}^*$, such that $\theta(ax) = b + \theta(x)$.
- First notice that if $x \notin \mathcal{W}$, then $ax \notin \mathcal{W}^*$:
  If $ax \in \mathcal{W}^*$, $st(ax) \in \mathcal{W}$. But $st(ax) = st(a)st(x) = 1.x = x$.
  In that case, $\theta(ax) = \theta(x) = 0$ by definition of the function θ.
- Suppose $x \in \mathcal{W}$.
  - If $ax \notin \mathcal{W}^*$, then $x \notin int(\mathcal{W})$: indeed, Theorem 3.11 tells us that $x \notin int(\mathcal{W})$ iff $\mu(x) \nsubseteq \mathcal{W}^*$, which is the case since $\mu(x) = \mu x$ contains the element ax which is not in $\mathcal{W}^*$.
    Note next that $\mathcal{U}^2 \subseteq int\mathcal{W}$:
    let $x \in \mathcal{U}^2$. Then $\mu.x \in (\mathcal{U}^3)^* \subseteq \mathcal{W}^*$, so, thanks to Theorem 3.11 again, we obtain that $\mathcal{U}^2 \subseteq int\mathcal{W}$.
    Consequently we conclude that $x \notin \mathcal{U}^2$, thus $\theta(x) = 0$. As $ax \notin \mathcal{W}^*$, $\theta(ax) = 0$ too.
  - Now suppose $ax \in \mathcal{W}^*$. Let $T_{y^{-1}}$ be the left translation in G where defined. As $\mathcal{U} \times \mathcal{W} \subseteq \Omega$, the function $\tau \circ T_{y^{-1}}$ is well defined and continuous on $\mathcal{W}$, for $y, y^{-1} \in \mathcal{U}$. The difference $|\tau \circ T_{y^{-1}}(ax) - \tau \circ T_{y^{-1}}(x)|$ then equals an

infinitesimal element of $\mathbb{R}^*$, which we call c. Therefore

$$\begin{aligned}\theta(ax) &= sup\{(1-\Delta(y))\tau(y^{-1}ax) : y \in \mathcal{U}\} \\ &= sup\{(1-\Delta(y))(c+\tau(y^{-1}x)) : y \in \mathcal{U}\}\end{aligned}$$

Whence $|\theta(ax)-\theta(x)| \leq c\, sup_{y\in\mathcal{U}}(1-\Delta(y)) \leq c$.
So $\theta(ax)-\theta(x)$ is infinitesimal since c is.

(3) This is clear from the definition: it suffices to consider the different cases for x to be in $\mathcal{U}$, $\mathcal{W}\setminus\mathcal{U}$, or in $G\setminus\mathcal{W}$. If $x \in \mathcal{U}$, as $1 \in \mathcal{U}$, we take $y_0 = 1$. Then $\Delta(y_0) = 0$ and $\tau(x) = \tau(y_0^{-1}x \leq \theta(x)$.

(4) Let $a \in Q$ such that $(a,x) \in \Omega$.
We want to show that $|\theta(ax)-\theta(x)| \leq \frac{1}{N}$.
- If $x \notin \mathcal{W}$, then $ax \notin \mathcal{U}^2$, otherwise we would have $x = a^{-1}ax \in \mathcal{U}^3 \subseteq \mathcal{W}$, a contradiction. Hence we have $\theta(ax) = \theta(x) = 0$.
- If $x \in \mathcal{W}\setminus\mathcal{U}^3$, then $ax \notin \mathcal{U}^2$ (the reason here is similar as the one used above). So $\theta(ax) = 0$. Since $x \notin \mathcal{U}^3 \Rightarrow x \notin \mathcal{U}^2$, $\theta(x) = 0$.
- If $x \in \mathcal{U}^3$, then $ax \in \mathcal{U}^4 \subseteq \mathcal{W}$.
We use the formula defining $\theta(x)$ and $\theta(ax)$. Let $y \in \mathcal{U}$. Hence $(a^{-1},y) \in \Omega$, and, following a previous remark,

$$|(1-\Delta(a^{-1}y)) - (1-\Delta(y))| \leq \frac{1}{N}$$

Furthermore, $y^{-1}ax$ is defined because $y^{-1}a = (a^{-1}y)^{-1}$ thanks to Corollary 1.27, the latter lie in $\mathcal{W}$, $x \in \mathcal{W}$, and we have supposed G globally inversible. Finally $y^{-1}ax = (a^{-1}y)^{-1}x$, thus

$$\begin{aligned}(*) &= |(1-\Delta(y))\tau(y^{-1}ax) - (1-\Delta(a^{-1}y)\tau((a^{-1}y)^{-1}x))| \\ &= |(1-\Delta(y)) - (1-\Delta(a^{-1}y)|.|\tau(y^{-1}ax)| \\ &\leq \frac{1}{N}.1\end{aligned}$$

Set

$$S := \{(1-\Delta(y))\tau(y^{-1}x) : y \in \mathcal{U}\}$$

and

$$S' := \{(1-\Delta(a^{-1}y)\tau((a^{-1}y)^{-1}x)) : y \in \mathcal{U}, a^{-1}y \in \mathcal{U}\}$$

Recall that $\Delta(x) = 1$ if $x \notin Q^N$. Hence $sup\, S = sup_{y\in Q^N}\{(1-\Delta(y))\tau(y^{-1}x)\}$, henceforth $sup\, S' = sup\, S$. Thus

$$\begin{aligned}|\theta(ax)-\theta(x)| &= |\theta(ax) - sup\, S| \\ &= |\theta(ax) - sup\, S'| \\ &= (*) \leq \frac{1}{N}\end{aligned}$$

□

1. The set C of continuous functions from G to $\mathbb{R}$ with compact support

Now let $f : G \to \mathbb{R}$. Recall that $supp(f) := \overline{\{x : f(x) \neq 0\}}$. We consider the set

$$C := \{f : G \to \mathbb{R} | f \text{ is continuous and } supp(f) \subseteq \mathcal{W}^2\}$$

C is a real vector space, which we equipp with the following norm :

$$||f|| := sup\{|f(x)| \, : x \in G\}$$

Next, for $f \in C$, $supp(f) \subseteq \mathcal{W}$ and $a \in \mathcal{W}$, we define a new function $a * f$ as follows:

$$(a * f)(x) = \begin{cases} f(a^{-1}x) & \text{if } x \in \mathcal{W}^2 \\ 0 & \text{otherwise} \end{cases}$$

Remarks :

- If $a^{-1}x \notin \mathcal{W} \cup \mathcal{W}^2$, $f(a^{-1}x) = 0$ because $f \in C$ and $supp(f) \subseteq \mathcal{W}$.
- Asking condition "$x \in \mathcal{W}^2$" instead of "$a^{-1}x \in \mathcal{W}$" (which would directly imply that $x \in \mathcal{W}^2$, as $x = aa^{-1}x$) allows to emphazise on the fact that $a * f \in C$.
- Let $f, g \in C$ such that $supp(f) \subseteq \mathcal{W}$ and $supp(g) \subseteq \mathcal{W}$. then $a * (f + g) = a * f + a * g$
- We will write af for $a * f$, which is not to be confused with the multiplication by a scalar.

Lemma 6.2. *Suppose $a, b \in \mathcal{W}$ and $f \in C$ are such that $supp(f) \subseteq \mathcal{W}$ and $supp(bf) \subseteq \mathcal{W}$. Then*

(i) $||af|| = ||f||$

(ii) $a(bf) = (ab)f$ *and* $||(ab)f - f|| \leq ||af - f|| + ||bf - f||$

Proof. (i)

$$\begin{aligned} ||af|| &= sup\{|f(a^{-1}x)| \, : x \in \mathcal{W}^2\} \\ &= sup\{|f(a^{-1}x)| \, : x \in \mathcal{W}^2, a^{-1}x \in \mathcal{W}\} \text{ because } supp(f) \subseteq \mathcal{W} \\ &= sup\{|f(y)| \, : y \in \mathcal{W}\} \\ &= ||f|| \end{aligned}$$

(ii) <u>First part</u>

Suppose $x \notin \mathcal{W}^2$. Thus, by definition, $((ab)f)(x) = 0$, and, since $a^{-1}x \notin \mathcal{W}$ and $supp(bf) \subseteq \mathcal{W}$, we get that $(a(bf))(x) = 0$.

Now suppose that $x \in \mathcal{W}^2$. Then

$$((ab)f)(x) = f((ab)^{-1}x) = f((b^{-1}a^{-1})x) \text{ and}$$

$$(a(bf))(x) = (bf)(a^{-1}x)$$

- If $a^{-1}x \notin \mathcal{W}^2$, then $b^{-1}a^{-1}x \notin \mathcal{W}$, else $bb^{-1}a^{-1}x \in \mathcal{W}^2$, so $b^{-1}a^{-1}x \notin supp(f)$. (Recall that $b^{-1}a^{-1}x$ is well defined and equals $(b^{-1}a^{-1})x$ as $\mathcal{W} \subseteq U_M$.) In that case, $((ab)f)(x) = f(b^{-1}a^{-1}x) = 0$, and $(a(bf))(x) = 0$.

- If $a^{-1}x \in \mathcal{W}^2$, then

$$(a(bf))(x) = (bf)(a^{-1}x) = f(b^{-1}(a^{-1}x)) = ((ab)f)(x)$$

Second part

$$\begin{aligned} ||(ab)f - f|| &= ||a(bf) - af + af - f|| \\ &\leq ||a(bf) - af|| + ||af - f|| \\ &\leq ||a(bf - f)|| + ||af - f|| = ||bf - f|| + ||af - f|| \end{aligned}$$

□

Back to the function θ which has been defined earlier, we see that

$$supp(\theta) \subseteq \mathcal{U}^2 \subseteq \mathcal{W}$$

Hence for any $a \in \mathcal{W}$, we can consider the function $a\theta$. We begin with an equicontinuity result.

LEMMA 6.3. *For each $\epsilon \in \mathbb{R}^{>0}$, there is a symmetric neighborhood V_ϵ of 1 in G, independent of Q, such that $V_\epsilon \subseteq \mathcal{U}$ and $||a\theta - \theta|| \leq \epsilon$ for all $a \in V_\epsilon$ (5).*

PROOF. Let $\epsilon \in \mathbb{R}^{>0}$. The function τ is continuous and $\mathcal{W}$ is compact, so τ is uniformly continuous on $\mathcal{W}$. Therefore there is a neighborhood O_ϵ of 1 in G such that for all $g, h \in \mathcal{W}$, if $gh^{-1} \in O_\epsilon^*$ then $|\tau(g) - \tau(h)| < \epsilon$. By Lemma 4.12 and because $\mathcal{U} \subseteq \mathcal{W}$, there exists a neighborhood V_ϵ of 1 in G such that for all $y \in \mathcal{U}$,

$$y^{-1}V_\epsilon y \subseteq O_\epsilon$$

We will show that this choice for V_ϵ does work. Let $a \in V_\epsilon$. We consider the difference $|(a\theta)(x) - \theta(x)|$ in different cases.

- If $x \notin \mathcal{W}^2$, then $(a\theta)(x) = \theta(x) = 0$.
- If $x \in \mathcal{W}^2 \setminus \mathcal{W}, \theta(x) = 0$.
  - If $a^{-1}x \notin \mathcal{W}, (a\theta)(x) = 0$.
  - If $a^{-1}x \in \mathcal{W}, a^{-1}x \notin \mathcal{U}^2$, else $x \in \mathcal{U}^3 \subseteq \mathcal{W}$. So in that case we also get $(a\theta)(x) = 0$, this time through definition of θ.
- If $x \in \mathcal{W} \setminus \mathcal{U}^3$, then $a^{-1}x \notin \mathcal{U}^2$, whence again we get that $(a\theta)(x) = \theta(x) = 0$.
- If $x \in \mathcal{U}^3$ (and thus $a^{-1}x \in \mathcal{U}^4 \subseteq \mathcal{W}$), then for all $y \in \mathcal{U}$, since

  $$y^{-1}a^{-1}x(y^{-1}x)^{-1} = y^{-1}a^{-1}y \in O_\epsilon \text{ because } a \in V_\epsilon$$

  uniform continuity of θ gives us that

  $$|\tau(y^{-1}a^{-1}x) - \tau(y^{-1}x)| < \epsilon$$

$$\begin{aligned} |(a\theta)(x) - \theta(x)| &= |sup_{y\in\mathcal{U}}(1 - \Delta(y))\tau(y^{-1}a^{-1}x) - sup_{y\in\mathcal{U}}(1 - \Delta(y))\tau(y^{-1}x)| \\ &\leq sup_{y\in\mathcal{U}}(1 - \Delta(y))|\tau(y^{-1}a^{-1}x) - \tau(y^{-1}x)| \\ &\leq \epsilon \end{aligned}$$

□

2. Local Haar measure

Set $C^+ := C \cap \{f : G \to \mathbb{R}^+\}$. Hypotheses on $\mathcal{U}$, $\mathcal{W}$ remain the same. We begin with a few definitions.

DEFINITION 6.4. *A linear application $F : C \to \mathbb{R}$ is called* ***a local Haar integral on*** *G if F is not the constant function equal to 0, if $F(C^+) \subseteq \mathbb{R}^+$ (F is* ***positive****), and if (****left invariance****) $F(af) = F(f)$ for all $f \in C$ such that supp$(f) \subseteq \mathcal{W}$ and $a \in \mathcal{W}$.*

In this section we will give an idea of how to construct such an integral, by looking for a real-valued function F^+ defined on C^+, satisfying the following conditions:

- F^+ is not the constant function equal to 0
- $F^+(C^+) \subseteq \mathbb{R}^+$
- $F^+(rf) = rF^+(f)$ for all $f \in C^+$ and all $r \in \mathbb{R}^+$
- $F^+(f+g) = F^+(f) + F^+(g)$ for all $f, g \in C^+$
- $F^+(af) = F^+(f)$ for all $f \in C^+$ such that $supp(f) \subseteq \mathcal{W}$ and $a \in \mathcal{W}$

Indeed, if there exists a local Haar integral F we can take $F^+ := F_{|C^+}$. Conversely, any F^+ satisfying conditions above is uniquely prolongable in a Haar integral in the following way:
For $f \in C$, set $f^+ : x \mapsto max(f(x), 0)$ and $f^- : x \mapsto max(-f(x), 0)$.
Then $f = f^+ - f^-$,with $f^+, f^- \in C^+$ and we put $F := F^+(f^+) - F^+(f^-)$.

As G is a T_2 and locally compact topological space, if F is a linear application from C to $\mathbb{R}$, the **Riesz representation theorem** asserts that there exists a Borel measure ξ on $\mathcal{W}^2$, which 'represents' F, i.e. such that for all $f \in C$, $F(f) = \int f d\xi$.
This measure is called a **local Haar measure**. By the left-invariance of the integral, the measure ξ is left-invariant in the sense that for any measurable subset $V \subseteq \mathcal{W}$, we have $\xi(aV) = \xi(V)$.

Now let $f, g \in C^+$ such that $supp(g) \neq \emptyset$. Let $x \in \mathcal{W}^2$.

- If $x \in supp(g)$, there exists $r_x \in \mathbb{R}^+$ such that $f(x) < r_x g(x)$.
- If $x \notin supp(g)$, let $y \in supp(g)$. Then one can write

$$y = yx^{-1}x = (xy^{-1})^{-1}x =: a_x x$$

  (Recall that $\mathcal{W} \subseteq U_M$ for a suitable $M \geq 6$). As above, there exists $r_x \in \mathbb{R}^+$ such that $f(x) < r_x g(a_x x) = r_x g(y)$.

$$\text{Set } U_x = \{x' \in \mathcal{W}^2 : f(x') < r_x g(x')\} \text{ for } x \in supp(g)$$

$$\text{and } U_x = \{x' \in \mathcal{W}^2 : f(x') < r_x g(a_x x')\} \text{ for } x \notin supp(g)$$

The U_x 's cover $\mathcal{W}^2$ which is compact, thus there is a finite subcover $U_1, \ldots, U_n$, i.e. there exists $r_1, \ldots, r_n > 0$, and $a_1, \ldots, a_n \in G$ such that for all $x \in U_i$, $f(x) < r_i g(a_i x)$.

In particular,

$$\exists r_1 > 0, \ldots, \exists r_n > 0, \exists a_1 \in G, \ldots, \exists a_n \in G, \forall x \in \mathcal{W}^2,$$

$$f(x) < \sum_{i=1}^{n} r_i g(a_i x) \quad (\bigstar)$$

Next consider the greatest lower bound of the $\sum_{i=1}^{n} r_i$'s corresponding to all possible majorations:

$$(f : g) := inf\{\sum_{i=1}^{n} r_i : (\bigstar)\}$$

Immediate properties:

(i)

$$(f : g) \geq \frac{sup\, f}{sup\, g}$$

Indeed, $f(x) < \sum_{i=1}^{n} r_i g(a_i x) \quad (\bigstar)$, thus $sup\, f \leq \sum_{i=1}^{n} r_i sup\, g$.

(ii) Let $f_1 \in C^+$ such that $supp(f_1) \neq \emptyset$. Then

$$(f : g) \leq (f : f_1).(f_1 : g)$$

(iii) $(f_1 : g) > 0$, so the above inequality can be written

$$\frac{(f : g)}{(f_1 : g)} \leq (f : f_1)$$

(iv) Suppose $supp(f) \neq \emptyset$. We thus also get $(f_1 : g) \leq (f_1 : f).(f : g)$, and

$$\frac{1}{(f_1 : f)} \leq \frac{(f : g)}{(f_1 : g)}$$

Finally,

$$\frac{1}{(f_1 : f)} \leq \frac{(f : g)}{(f_1 : g)} \leq (f : f_1)$$

So if we fix $f, f_1 \in C^+$ such that $supp(f) \neq \emptyset$ and $supp(f_1) \neq \emptyset$, we can define an associate 'interval' :

$$I_f := \left[\frac{1}{(f_1 : f)}; (f : f_1)\right] \subseteq \mathbb{R}^+ \setminus \{0\}$$

if f is not the constant function equal to 0 ; otherwise if $f = 0$, set $I_f := \{0\}$. Then the product

$$P := \prod_{f \in C^+} I_f$$

is compact by Tychonoff's Theorem. If $g \in C^+$ is not the constant function equal to 0 and $supp(g) \neq \emptyset$, $\frac{(f:g)}{(f_1:g)} \in I_f$ for all f; meaning that to each such function g, we associate a point p_g in P. Let $V \subseteq \mathcal{W}$ be a neighborhood of 1 in G, and let C_V^+ be the set of elements of C^+ the support of which is nonempty and included in V.

Since G is locally compact, $C_V^+ \neq \emptyset$ thanks to Urysohn Lemma 1.3. Let

$$F_V := \overline{\{p_g : g \in C_V^+\}}$$

Next consider the family $(F_V)_{V \subseteq \mathcal{W} \text{ is a neighborhood of } 1}$. This family has the finite intersection property, hence by compacity of P there is a point p in the intersection of all F_V. This point can be written $p = (r_f)_{f \in C^+}$. Let $\int f := r_f$. This function, $\int$, satisfies conditions mentioned above. It is also unique, up to multiplication by a positive real constant. For a proof, see [**13**], or [**8**].

3. Lemmas

Fix a continuous function $\tau_1 : G \to [0;1]$ such that

$$\tau_1 : \begin{array}{ccc} x & \mapsto & 1 \quad \text{if } x \in \mathcal{U}^2 \\ x & \mapsto & 0 \quad \text{if } x \in G \setminus \mathcal{U}^3 \end{array}$$

As for the function τ previously defined, it will be important later that τ_1 depends only on $\mathcal{U}$ and not Q. Notice that $0 \leq \theta \leq \tau_1$, since $supp(\theta) \subseteq \mathcal{U}^2$. Take the unique Haar measure ξ such that $\int \tau_1(x) d\xi(x) = 1$: since $\tau_1 \in C$, there is $b \in \mathbb{R}$ such that $\int \tau_1(x) d\xi(x) = b$, and by left-invariance we can suppose that $\int \tau_1(x) d\xi(x) = 1$.

We thus have

$$(\triangle) \;\; 0 \leq \int \theta(x) d\xi(x) \leq 1$$

By Lemma 6.3, there is an open neighborhood $V := V_{\frac{1}{2}} \subseteq \mathcal{U}$ of 1 in G, independent of Q, and such that:

(6)

$$\forall a \in V, ||a\theta - \theta|| \leq \frac{1}{2}$$

Since $||\theta|| \leq 1$,it is also true for θ^2: $||a\theta^2 - \theta^2|| \leq \frac{1}{2}$ on V.
Hence, for $a \in V$,

$$(a\theta^2(x) - \theta^2(x))_{|x=a} = \theta^2(a^{-1}a) - \theta^2(a) = 1 - \theta^2(a) \leq \frac{1}{2}$$

Thus $\theta^2(a) \geq 1 - \frac{1}{2}$ for all $a \in V$.

Now let $\Phi := \Phi_Q : G \to \mathbb{R}$ defined as follows:

$$\Phi : \begin{array}{ccc} x & \mapsto & \int \theta(xu)\theta(u) d\xi(u) \quad \text{if } x \in \mathcal{W} \\ x & \mapsto & 0 \quad \text{if } x \in G \setminus \mathcal{W} \end{array}$$

Φ is continuous as $\mathcal{W}$ is compact; moreover:

(7) $supp(\Phi) \subseteq \mathcal{U}^4 \subseteq \mathcal{W}$
(8) $\Phi(1) \geq \frac{\xi(V)}{2}$
(9) If $a \in Q$, then $||a\Phi - \Phi|| \leq \frac{1}{N+1}$

(10) For all $a \in \mathcal{W}$, $||a\Phi - \Phi|| \leq ||a\theta - \theta||$

PROOF. (7) By Lemma 6.1, item (1),$supp(\theta) \subseteq \mathcal{U}^2$.

(8) $\Phi(1) = \int \theta^2(u) d\xi(u) \geq \frac{1}{2} \int_V d\xi(u) = \frac{\xi(V)}{2}$

(9) To verify (9) and (10), one needs to distinct cases such as in the proof of Lemma 6.1; nevertheless, there is only one case where the first clauseof the definition applies for both $a\Phi$ and Φ; in that case:

$$\begin{aligned}
||a\Phi - \Phi|| &= ||\Phi(a^{-1}x) - \Phi(x)|| \\
&= ||\int \theta(a^{-1}xu)\theta(u)d\xi(u) - \int \theta(xu)\theta(u)d\xi(u)|| \\
&= ||\int (\theta(a^{-1}xu) - \theta(xu))\theta(u)d\xi(u)|| \ (\blacktriangle) \\
&\leq ||\int \frac{\theta(u)d\xi(u)}{N+1}|| \text{ by Lemma 6.1 item (4)} \\
&\leq \frac{1}{N+1} \text{ by } (\triangle)
\end{aligned}$$

where $N = ord_{\mathcal{U}}(Q)$.

(10) Use (6) and ($\blacktriangle$)

□

By (7), $supp(a\Phi - \Phi) \subseteq \mathcal{U}^5 \subseteq \mathcal{W}$ for all $a \in \mathcal{W}$, thus for all $b \in \mathcal{W}$, the function $b(a\Phi - \Phi))$ is defined. The next two Lemmas are the local versions of the Gleason-Yamabe Lemmas.

LEMMA 6.5. *Let $c, \epsilon \in \mathbb{R}^+ \setminus \{0\}$. Then there is a neighborhood $U = U_{c,\epsilon} \subseteq \mathcal{U}$ of 1 in G, independent of Q, such that for all $a \in Q$, $b \in U$, and $m \leq cN$,*

$$||b.m(a\Phi - \Phi) - m(a\Phi - \Phi)|| \leq \epsilon$$

and

$$||m(a\Phi - \Phi)|| \leq c$$

PROOF. Let $a \in Q$, $b \in \mathcal{U}$. We have $supp(b.m(a\Phi - \Phi) - m(a\Phi - \Phi)) \subseteq \mathcal{U}^6$. Let $x \in \mathcal{U}^6$ and put $y := b^{-1}x$. Hence

$$(a\Phi - \Phi)(x) = \int \left[\theta(a^{-1}xu) - \theta(xu)\right] \theta(u) d\xi(u)$$

and

$$\begin{aligned}
b(a\Phi - \Phi)(x) &= (a\Phi - \Phi)(y) \\
&= \int \left[\theta(a^{-1}yu) - \theta(yu)\right] \theta(u) d\xi(u)
\end{aligned}$$

Recall that we are working with the assumption that $\mathcal{U}^P \subseteq \mathcal{W}$ for a suitably large P, which can thus be chosen such that $y^{-1}x \in \mathcal{W}$. The latter permits to use left-invariance of the integral in order to replace u by $x^{-1}yu$ in the function of u integrated in the first identity: $\int y^{-1}xf = \int f$. This gives:

$$\begin{aligned}(a\Phi - \Phi)(x) &= \int \left[\theta(a^{-1}xx^{-1}yu) - \theta(xx^{-1}yu)\right]\theta(x^{-1}yu)d\xi(x^{-1}yu)\\ &= \int \left[\theta(a^{-1}yu) - \theta(yu)\right]\theta(x^{-1}yu)d\xi(u)\end{aligned}$$

Then

$$\left[b.(a\Phi - \Phi) - (a\Phi - \Phi)\right](x) = \int \left[(a\theta - \theta)(yu)\right]\left[(\theta - y^{-1}x\theta)(u)\right]d\xi(u)$$

By Lemma 6.3, it is possible to choose the neighborhood $U_{c,\epsilon}$ small enough to satisfy: for all $b \in U_{c,\epsilon}$ and $x \in \mathcal{U}^6$ we have $y^{-1}x \in \mathcal{U}$ and

$$||\theta - y^{-1}x\theta|| < \frac{\epsilon}{c\xi(\mathcal{U}^6)}$$

Then

$$\left[b.(a\Phi - \Phi) - (a\Phi - \Phi)\right](x) \leq \frac{1}{N}\frac{\epsilon}{c\xi(\mathcal{U}^6)}.\xi(\mathcal{U}^6) \leq \frac{\epsilon}{cN} \leq \epsilon$$

For $1 < m \leq cN$, it is then clear. □

LEMMA 6.6. *With $c, \epsilon \in \mathbb{R}^+ \setminus \{0\}$, let $U := U_{c,\epsilon}$ be as in the previous Lemma and let $a \in Q$ and m, n be such that $m \leq cN$, $n > 0$, a^n is defined, and $a^i \in U$ for $i \in \{0, \ldots, n\}$. Then*

$$||(\frac{m}{n})(a^n\Phi - \Phi) - m(a\Phi - \Phi)|| \leq \epsilon$$

PROOF. We first show that $m(a^n\Phi - \Phi) = \sum_{i=0}^{n-1} a^i m(a\Phi - \Phi)$ by induction on n:
This is clear when $n = 0$ and $n = 1$. Suppose this is true at rank $n-1$, for $n \geq 1$. Then

$$\begin{aligned}m(a^n\Phi - \Phi) &= m(a^n\Phi - a^{n-1}\Phi + a^{n-1}\Phi - \Phi)\\ &= m(a^n\Phi - a^{n-1}\Phi) + m(a^{n-1}\Phi - \Phi))\\ &= ma^{n-1}(a\Phi - \Phi) + \sum_{i=0}^{n-2} a^i m(a\Phi - \Phi)\\ &= a^{n-1}m(a\Phi - \Phi) + \sum_{i=0}^{n-2} a^i m(a\Phi - \Phi)\\ &= \sum_{i=0}^{n-1} a^i m(a\Phi - \Phi)\end{aligned}$$

Hence

$$m(a^n\Phi - \Phi) - nm(a\Phi - \Phi) = \sum_{i=0}^{n-1}(a^i m(a\Phi - \Phi) - m(a\Phi - \Phi))$$

Let $i \in \{1, \ldots, n\}$. Then a^i is defined and belongs to U, thus by Lemma 6.5

$$||a^i m(a\Phi - \Phi) - m(a\Phi - \Phi)|| \leq \epsilon$$

whence by summing:

$$||m(a^n\Phi - \Phi) - mn(a\Phi - \Phi)|| \leq \epsilon n$$

□

Now we suppose that Q is a symmetric internal subset of μ such that $1 \in Q$ and $N := ord_{\mathcal{U}}(Q) \in \mathbb{N}^*$. Notice that $N > \mathbb{N}$.

The constructions made in this section transfer to the nonstandard setting and yield internal continuous functions $\theta : G^* \to [0;1]^*$ and $\Phi : G^* \to \mathbb{R}^*$ satisfying the internal versions of (1)-(10) and Lemmas 6.5 and 6.6.

Lemma 6.7. *Suppose $a \in Q$, $\nu = O(N)$, a^ν is defined and $a^\sigma \in \mu$ for all $\sigma \leq \nu$. Then $||\nu(a\Phi - \Phi)||$ is infinitesimal.*

Proof. Let $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Let $c \in \mathbb{R}$, $c > 0$ be such that $\nu \leq cN$. By Lemma 6.6 with $m = n = \nu$, we get:

$$||(a^\nu\Phi - \Phi) - \nu(a\Phi - \Phi)|| \leq \epsilon$$

(10) and Lemma 6.3 give us:

$$||a^\nu\Phi - \Phi|| \leq ||a^\nu\theta - \theta|| < \epsilon$$

hence $||\nu(a\Phi - \Phi)|| \leq 2\epsilon$. □

CHAPTER 7

Group structure on $L(G)$

In this section, we see some consequences of the Gleason-Yamabe Lemmas allowing to put a group structure on $L(G)$. We begin with the local versions, given by I.Goldbring, of the monadic forms of Gleason Yamabe Lemmas given by J.Hirschfeld in [**7**]. The results are statements about growth rates of powers of infinitesimals. Lemmas 7.1 to 7.4, Theorems 7.5 and 7.6 are from [**5**], we give more detailed proofs.

1. Local monadic form of the Gleason Yamabe Lemmas

LEMMA 7.1. *Let $U \subseteq U_2$ be a compact symmetric neighborhood of* 1. *Suppose $Q^\nu \nsubseteq \mu$ for some $\nu = o(ord_U(Q))$. Then U contains a nontrivial connected subgroup of G.*

PROOF. Like previously seen in Lemma 4.8, the set

$$G_U(Q) = \{st(a) : a \in Q^\nu \text{ for some } \nu = o(ord_U(Q))\}$$

is a union of connected subsets of U, each containing 1. It is thus a connected subset of U. It is closed under taking inverses by Lemma 1.28; and it is closed under products:
let a, b be such that $st(a), st(b) \in G_U(Q)$. There are $\eta, \theta = o(ord_U(Q))$ such that $a \in Q^\eta$, $b \in Q^\theta$. Hence $\eta + \theta = o(ord_U(Q))$, $a.b$ is defined and in $Q^{\eta+\theta} \subseteq U^* \subseteq G^*_{ns}$. Thus we can consider $st(a.b)$ which is in $G_U(Q)$ and equals $st(a).st(b)$. □

From now on, we suppose that G is NSS.

LEMMA 7.2. *Suppose $\nu \in \mathbb{N}^* \setminus \mathbb{N}$ and $a_1, \ldots, a_\nu$ is a hyperfinite sequence such that $a_i \in G^o(\nu)$ for all $i \in \{1, \ldots, \nu\}$. Let $Q := \{1, a_1, \ldots, a_\nu, a_1^{-1}, \ldots, a_\nu^{-1}\}$. Then Q^ν is defined and $Q^\nu \subseteq \mu$.*

PROOF. (1) We begin by showing that if Q^ν is defined, then $Q^\nu \subseteq \mu$. Suppose not, i.e. suppose that $Q^\nu \nsubseteq \mu$. Then there is a neighborhood U of 1 in G such that Q^ν, hence $Q^{\nu+1}$ is not included in U^*. It is possible to choose U symmetric and compact . Note that $Q^{\nu+1} \nsubseteq U^*$ implies that $ord_U(Q) \leq \nu$. Then fix a set $\mathcal{W}$ as in the setting of the Gleason Yamabe lemmas, and take U small enough to have $U \subseteq \mathcal{W}$ and $U^P \subseteq \mathcal{W}$. Recall $\mathcal{W}$ is symmetric, compact and that $\mathcal{W} \subseteq U_M$.

If necessary, by decreasing ν, and Q accordingly, it is possible to obtain that $ord_U(Q) = \nu$: let $ord_U(Q) =: \nu_0 \leq \nu$, and consider

$$Q_{\nu_0} := \{1, a_1, \ldots, a_{\nu_0}, a_1^{-1}, \ldots, a_{\nu_0}^{-1}\}$$

By Lemma 3.22, ν_0 is infinite, and thus $a_1, \ldots, a_{\nu_0}$ is a hyperfinite sequence. Then set $\nu_1 := ord_U(Q_{\nu_0})$, and so on: we construct a decreasing sequence $(\nu_i, Q_{\nu_i})_{i \in J}$. Suppose the set J is not finite; then by Lemma 3.4, as the internal set $A := \{\nu_0, \ldots, \nu_i, \ldots\}$ contains arbitrarily small positive infinite elements, it contains a finite element, which is a contradiction. We can thus choose i maximal such that ν_i is minimal. Then $\nu_{i+1} := ord_U(Q_{\nu_i}) = \nu_i$.

Claim: $Q^\nu \nsubseteq \mu \Rightarrow Q_{\nu_i}^{\nu_i} \nsubseteq \mu$.
Suppose $Q_{\nu_i}^{\nu_i} \subseteq \mu$ and $Q^\nu \nsubseteq \mu$. Then there is $b \in Q \setminus Q_{\nu_i}$ such that $(a_1 \ldots a_{\nu_i})b$ (which is well defined because it belongs to $Q^{\nu+1}$) is not in μ. This contradicts the fact that μ is a subgroup, since $b \in \mu$ and $a_1 \ldots a_{\nu_i} \in \mu$.
Let $\nu := \nu_i$. We will consider two cases:

- First, suppose that $Q^i \subseteq \mu$ for all $i = o(\nu)$. As $Q^\nu \nsubseteq \mu$, we can take $b \in Q^\nu$ such that $st(b) \neq 1$. We then choose $\mathcal{U}$ as in the setting of the Gleason-Yamabe lemmas so that $\mathcal{U} \subseteq U$ and $st(b) \notin \mathcal{U}^4$: indeed we can find open neighborhoods of 1 O_i and O_j such that $O_i \cap st(b).O_j = \emptyset$ (G is homogeneous and T_2), and then by regularity and by a consequence of Lemma 1.25, there is a compact symmetric neighborhood $\mathcal{U} \subseteq U$ of 1 such that $\mathcal{U}^4 \subseteq O_i$, henceforth $st(b) \notin \mathcal{U}^4$.

  Now set $\eta := ord_{\mathcal{U}}(Q)$. Then $\nu = O(\eta)$, otherwise $\eta = o(\nu)$ which would imply that $Q^\eta \subseteq \mu$ by our hypothesis, contradicting the fact that $\eta := ord_{\mathcal{U}}(Q)$.

  The Transfer Principle allows to extend the function constructed in the previous section to the internally continuous function $\Phi = \Phi_Q : G^* \to \mathbb{R}^*$, hence satisfying the internal versions of all the properties and lemmas from that section. In particular, $\Phi(b^{-1}) = 0$, since $supp(\Phi) \subseteq \mathcal{U}^4$, and $st(b) \notin \mathcal{U}^4$, henceforth $st(b^{-1}) \notin \mathcal{U}^4$. Moreover, $\Phi(1) \geq \frac{\xi(V)}{2} \in \mathbb{R}$ by property (8), hence $\Phi(1)$ is not infinitesimal. Consequently, $||b\Phi - \Phi||$ is not infinitesimal, since

$$sup\{|\Phi(b^{-1}x) - \Phi(x)| : x \in G\} \geq |\Phi(b^{-1}.1) - \Phi(1)|$$

  However, if we take a hyperfinite sequence $b_1, \ldots, b_\nu$ from Q such that $b = b_1 \ldots b_\nu$, we get, by the Transfer Principle and

Lemma 6.2:

$$||b\Phi - \Phi|| \leq \sum_{i=1}^{\nu} ||b_i\Phi - \Phi||$$

Let $b_0 \in \{b_1, \ldots, b_\nu\}$ be such that

$max\{||b_i\Phi - \Phi|| : i \in \{1, \ldots, \nu\}\} = ||b_0\Phi - \Phi||$

Then $\sum_{i=1}^{\nu} ||b_i\Phi - \Phi|| \leq \nu||b_0\Phi - \Phi||$ which is infinitesimal by Lemma 6.7, contradicting the fact that $||b\Phi - \Phi||$ is not infinitesimal.

- Now suppose that $Q^i \nsubseteq \mu$ for some $i = o(\nu)$. Then the compact subgroup of G defined in Lemma 7.1

  $G_U(Q) := \{st(a) : a \in Q^\theta \text{ for some } \theta = o(ord_U(Q))\}$

  is nontrivial (and contained in U), which contradicts our assumption that G is NSS.

(2) Next we want to show that Q^ν is defined. Let $E \subseteq \mu$ an internal set.

Claim: Let $c_1, \ldots, c_\eta$ be an internal sequence such that for all $i, j \in \{1, \ldots, \eta\}$, c_i^j is defined and $c_i^j \in E$. Let

$$R_\eta = \{1, c_1, \ldots, c_\eta, c_1^{-1}, \ldots, c_\eta^{-1}\}$$

Then R_η^η is defined.

The proof is made by internal induction on η. The case $\eta = 1$ is obvious. Suppose inductively that the claim holds for a given η, and let $c_1, \ldots, c_{\eta+1}$ be an internal sequence such that for all $i, j \in \{1, \ldots, \eta+1\}$, c_i^j is defined and $c_i^j \in E$. Let $R_{\eta+1} = \{1, c_1, \ldots, c_{\eta+1}, c_1^{-1}, \ldots, c_{\eta+1}^{-1}\}$, and $d_1, \ldots, d_{\eta+1} \in R_{\eta+1}$. The product $d_1 \ldots d_\eta$ is defined by induction hypothesis. By the first part of the proof, it is thus infinitesimal. Similarly, $d_1 \ldots d_i$ and $d_i \ldots d_{\eta+1}$ are defined and infinitesimals for all $i \in \{2, \ldots, \eta\}$; hence $d_1 \ldots d_{\eta+1}$ is defined by Lemma 1.28. This shows that $R_{\eta+1}^{\eta+1}$ is defined.

Now let $a_1, \ldots, a_\nu$ be an internal sequence as in the statement of the lemma. Taking $E = \{a_i^j : i, j \in \{1, \ldots, \nu\}\}$, we obtain that Q^ν is defined.

□

LEMMA 7.3. *Suppose U is a compact symmetric neighborhood of* 1 *in G with $U \subseteq U_2$. Let $\nu > \mathbb{N}$ be such that for all $i \in \{1, \ldots, \nu\}$, a^i and b^i are defined and $a^i \in U^*$, $b^i \in \mu$. Then for all $i \in \{1, \ldots, \nu\}$, we have that $(ab)^i$ is defined and $(ab)^i \sim a^i$.*

PROOF. Let $i \in \{1, \ldots, \nu\}$. Then $(a^i, b) \in \Omega^*$ and, using Lemma 3.21, $st(a^i b) = st(a^i)st(b) = st(a^i)$, whence $a^i b \sim a^i$. By hypothesis, a^i is defined, thus a^{-i} is defined by Lemma 1.28. Recall that we supposed G to be locally compact, hence, applying Theorem 3.15, we can consider the standard parts of the above elements. Furthermore $a^{-i} = (a^i)^{-1}$, and $st(a^{-i}) = st(a^i)^{-1}$, so $(st(a^i), st(a^{-i})) \in \Omega$, and we finally get $(a^i b, a^{-i}) \in \Omega^*$.

Similarly, $(a^i, ba^{-i}) \in \Omega^*$. As $(a^i b, a^{-i}) \in \Omega^*$ and $(a^i, ba^{-i}) \in \Omega^*$ for all $i \in \{1, \dots, \nu\}$, Lemma 1.28 allows to define the element $b_i := a^i b a^{-i}$. Furthermore, using same technical tools, one can define the element $b_i^j := a^i b^j a^{-i}$ for all $i, j \in \{1, \dots, \nu\}$. Observe that $b_i \in \mu$ for all $i \in \{1, \dots, \nu\}$.

Claim 1: For all $i, j \in \{1, \dots, \nu\}$, b_i^j is defined and $b_i^j = a^i b^j a^{-i}$.

This claim is shown by internal induction on j. The case $j = 1$ has already been proven. Suppose the assertion is true for all $j' \in \{1, \dots, j\}$ and that $j + 1 \leq \nu$. Let $k, l \in \{1, \dots, j\}$ be such that $k + l = j + 1$. By the induction hypothesis, we have that b_i^k is defined and $b_i^k = a^i b^k a^{-i}$. As $b^k \in \mu$, $b_i^k \in \mu$. For the same reasons b_i^l is defined and $b_i^l \in \mu$, therefore $(b_i^k, b_i^l) \in \Omega^*$. Using Lemma 1.28, we obtain that b_i^{j+1} is defined. Then, since $b_i^j \in \mu$, we have

$$\begin{aligned} b_i^{j+1} &= b_i^j . b_i \\ &= (a^i b^j a^{-i}).(a^i b a^{-i}) \\ &= ((a^i b^j a^{-i}).(a^i)).(b a^{-i}) \\ &= (a^i b^j).(b a^{-i}) \\ &= ((a^i b^j).b) a^{-i}) \\ &= a^i b^{j+1} a^{-i} \end{aligned}$$

We thus have proved Claim 1, from which it follows that $b_i^j \in \mu$ for all $i, j \in \{1, \dots, \nu\}$, meaning that $b_i \in G^o(\nu)$ by Lemma 4.2. Consequently, by Lemma 7.2, $b_1 \dots b_i$ is defined and $b_1 \dots b_i \in \mu$ for all $i \in \{1, \dots, \nu\}$. To end the proof of the lemma, we need the following statement:

Claim 2: For all $i \in \{1, \dots, \nu\}$, $(ab)^i$ is defined and $(ab)^i = (b_1 \dots b_i) a^i$.

Once again we use internal induction, on i. For $i = 1$, this is clearly true: indeed $ab = (aba^{-1})a$. So suppose the assertion holds for all $j \in \mathbb{N}^*$ with $j \leq i$ and $i + 1 \leq \nu$. To show that $(ab)^{i+1}$ is defined, it suffices, thanks to Lemma 1.28, to check that $((ab)^k, (ab)^l) \in \Omega^*$ for all $k, l \in \{1, \dots, i\}$ with $k + l = i + 1$. By the induction hypothesis,

$$(ab)^k = (b_1 \dots b_k) a^k \sim a^k \text{ and } (ab)^l = (b_1 \dots b_l) a^l \sim a^l$$

so the desired result is given by the fact that $(a^k, a^l) \in U^* \times U^*$. Next note that

$$\begin{aligned} a^{i+1}b &= (a^{i+1}.b).(a^{-i-1}a^{i+1}) \\ &= (a^{i+1}ba^{-i-1})a^{i+1} \\ &= b_{i+1}a^{i+1} \end{aligned}$$

Now, using the induction hypothesis, we obtain:

$$\begin{aligned} (ab)^{i+1} &= ((ab)^i).(ab) \\ &= ((b_1 \dots b_i)a^i).(ab) \\ &= (b_1 \dots b_i).(a^i(ab)) \\ &= (b_1 \dots b_i).(a^{i+1}b) \\ &= (b_1 \dots b_i).(b_{i+1}a^{i+1}) \\ &= (b_1 \dots b_{i+1})a^{i+1} \end{aligned}$$

□

Lemma 7.4. *Let $\nu > \mathbb{N}$ and $a \in G(\nu)$ be such that a^ν is defined and $a^i \in G^*_{ns}$ for all $i \in \{1, \dots, \nu\}$. Suppose also that $b \in \mu$ is such that b^ν is defined and $b^i \in G^*_{ns}$ and $a^i \sim b^i$ for all $i \in \{1, \dots, \nu\}$. Then $a^{-1}b \in G^o(\nu)$.*

Proof. The proof begin with explaining a series of reductions that it is possible to make without loss of generality. First, notice that $a^{-1}b \in G(\nu)$: Let $i = o(\nu)$. Since $a \in G(\nu)$, so is a^{-1}, hence $(a^{-1})^i = a^{-i}$ is defined and belongs to μ. As $b^i \sim a^i$, b^i is also in μ. Consequently, using Lemma 7.3, we obtain that $(a^{-1}b)^i$ is defined and $(a^{-1}b)^i \sim a^{-i}$, i.e. $a^{-1}b \in G(\nu)$.
Let V be a compact symmetric neighborhood of 1 in G such that $V \subseteq U_2$. Then since $\nu = O(ord_V(a^{-1}b))$ (otherwise $ord_V(a^{-1}b) = o(\nu)$ and $(a^{-1}b)^{ord_V(a^{-1}b)} \in \mu$, the latter being a group, we have a contradiction), if $\nu > ord_V(a^{-1}b)$ we may replace ν with $ord_V(a^{-1}b)$ and assume that $(a^{-1}b)^i$ is defined and belongs to G^*_{ns} for all $i \in \{1, \dots, \nu\}$.Let $Q = \{1, a, a^{-1}, b, b^{-1}\}$. Suppose, towards a contradiction, that Q^ν is not defined, and let $\eta \in \mathbb{N}^* \setminus \mathbb{N}$ be the largest element of $\mathbb{N}^*$ for which Q^η is defined. If $\eta = o(\nu)$, Lemma 7.2 implies that $Q^i \subseteq \mu$ for all $i \in \{1, \dots, \eta\}$. Henceforth Lemma 1.28 implies that $Q^{\eta+1}$ is defined, meaning that we have a contradiction. Thus $\nu = O(\eta)$; therefore, by replacing ν by η if necessary, we can assume Q^ν is defined.

Next observe that if $a \in G^o(\nu)$, then Lemma 7.3 gives us that $(a^{-1}b)^i$ is defined and $(a^{-1}b)^i \sim a^{-i} \in \mu$ for all $i \in \{1, \dots, \nu\}$, i.e. the proof is done. So now we suppose that $a \notin G^o(\nu)$ and by replacing ν by a smaller element of its archimedean class, we may assume as well that $a^\nu \notin \mu$.

Then we suppose, in order to yield a contradiction, that there is $j \in \{1, \dots, \nu\}$ such that $(a^{-1}b)^j \notin \mu$. Note that $\nu = O(j)$. Using properties of separation, we can choose compact symmetric neighborhoods $\mathcal{U}$ and $\mathcal{W}$ of 1 in G as in the previous section and so that

$$a^\nu \in \mathcal{W}^* \setminus \mathcal{U}^* \text{ and } (a^{-1}b)^j \in \mathcal{W}^* \setminus (\mathcal{U}^*)^4$$

Let $\sigma = ord_{\mathcal{U}}(Q)$. By Lemma 7.2, $\nu = O(\sigma)$, otherwise, if we had $\sigma = o(\nu)$, we would have that $Q^i \subseteq \mu$ for all $i \in \{1, \dots, \sigma\}$, and, like above, using Lemma 1.28, $Q^{\sigma+1}$ would be defined, a contradiction.

Let $\phi = \phi_Q : G^* \to \mathbb{R}^*$ be the internally continuous function constructed in the previous section. Then, as $(a^{-1}b)^j \notin (\mathcal{U}^*)^4 \supseteq supp(\phi)^*$, we get that $\phi((a^{-1}b)^j) = 0$. For reasons similar to those seen in the proof of Lemma 7.2, $\epsilon := \phi(1) > 0$ is not infinitesimal. We also have

$$\begin{aligned}
\epsilon = \phi(1) &= |\phi((a^{-1}b)^j) - \phi(1)| \\
&\leq ||(b^{-1}a)^j\phi - \phi|| \\
\text{(Lemma 6.2)} \qquad &\leq j||(b^{-1}a)\phi - \phi|| \\
\text{(left invariance)} \qquad &= j||b((b^{-1}a)\phi - \phi)|| \\
&= j||a\phi - b\phi|| \\
&= ||j(a\phi - \phi) - j(b\phi - \phi)||
\end{aligned}$$

To obtain a contradiction, we will show that $||j(a\phi - \phi) - j(b\phi - \phi)|| < \epsilon$. By Lemma 6.6, there is a compact symmetric neighborhood $U \subseteq \mathcal{U}$ of 1 in G such that for all $k > 0$ in $\mathbb{N}^*$, if $a^i \in U^*$ and $b^i \in U^*$ for all $i \in \{1, \dots, k\}$, then

$$||\frac{j}{k}(a^k\phi - \phi) - j(a\phi - \phi)|| < \frac{\epsilon}{3}$$

$$||\frac{j}{k}(b^k\phi - \phi) - j(b\phi - \phi)|| < \frac{\epsilon}{3}$$

These equalities work if $k = min\{ord_U(a), ord_U(b)\}$. For such an element k, $\nu = O(k)$, hence $j = O(k)$. Since $k < \nu$, $a^k \sim b^k$, so by continuity

$$||\frac{j}{k}(a^k\phi - \phi) - \frac{j}{k}(b^k\phi - \phi)|| = \frac{j}{k}||a^k\phi - b^k\phi|| \sim 0$$

The combination of the three inequalities gives

$$||j(a\phi - \phi) - j(b\phi - \phi)|| < \epsilon$$

which is a contradiction. □

2. Addition

THEOREM 7.5. *Suppose $\sigma > \mathbb{N}$. Then*

(1) $G(\sigma)$ and $G^o(\sigma)$ are normal subgroups of μ.
(2) If $a \in G(\sigma)$ and $b \in \mu$, then $aba^{-1}b^{-1} \in G^o(\sigma)$
(3) $G(\sigma)/G^o(\sigma)$ is abelian.

PROOF. (2) clearly implies (3), hence we only need to show (1) and (2).

(1) Let $a, b \in G(\sigma)$ (resp. $a, b \in G^o(\sigma)$). By Lemma 1.28, $a^{-1} \in G(\sigma)$ (resp. $G^o(\sigma)$), and then, by Lemma 7.3, $(a^{-1}b)^\nu$ is defined and $(a^{-1}b)^\nu \sim a^\nu \in \mu$ for all $\nu = o(\sigma)$ (resp. $\nu = O(\sigma)$).

By Lemma 5.7, $G(\sigma)$ is a normal subgroup of μ. Let $a \in G^o(\sigma)$ and $b \in \mu$. Let us show by internal induction on $\eta \leq \sigma$ that $(bab^{-1})^\eta$ is defined and equal to $ba^\eta b^{-1}$, as it will show in particular that $bab^{-1} \in G^o(\sigma)$: suppose the assertion is true for all $\nu \leq \eta$. Then

$$\begin{aligned}(bab^{-1})^{\eta+1} &= (bab^{-1})^\eta(bab^{-1})\\ &= ba^\eta b^{-1}.bab^{-1}\\ &= ba^{\eta+1}b^{-1}\end{aligned}$$

(2) Let $a \in G(\sigma)$ and $b \in \mu$. We know that $ba^{-1}b^{-1} \in G(\sigma)$. Let $U \subseteq U_2$ be a compact symmetric neighborhood of 1. As $ord_U(a) > \mathbb{N}$ and $ord_U(ba^{-1}b^{-1}) > \mathbb{N}$, it is possible to choose $\tau \in \{1, \ldots, \sigma\}$ such that $\sigma = O(\tau)$, a^τ and $(ba^{-1}b^{-1})^\tau$ are defined, and $a^i \in U^*$ for $i \in \{1, \ldots, \tau\}$. Like above, it is then possible to prove by internal induction that for $i \in \{1, \ldots, \tau\}$, $(ba^{-1}b^{-1})^i = ba^{-i}b^{-1}$, hence $(ba^{-1}b^{-1})^i \in \mu$, i.e., using Lemma 4.2, $ba^{-1}b^{-1} \in G^o(\sigma)$. In particular, $(ba^{-1}b^{-1})^i \in \mu$ for all $i \in \{1, \ldots, \sigma\}$. Henceforth, for those i's, $a^{-i} \sim (ba^{-1}b^{-1})^i$ and, with Lemma 7.4 applied to a^{-1} and $ba^{-1}b^{-1}$, we obtain $aba^{-1}b^{-1} \in G^o(\sigma)$.

□

Remark : if $\mathbb{X} \in L(G)$ and $\sigma > \mathbb{N}$, then $\mathbb{X}(\frac{1}{\sigma}) \in G(\sigma)$. This is because if $i = o(\sigma)$, then $(\mathbb{X}(\frac{1}{\sigma}))^i = \mathbb{X}(\frac{i}{\sigma}) \in \mu$.

THEOREM 7.6. *The map $S : L(G) \to G(\sigma)/G^o(\sigma)$ defined by*

$$S(\mathbb{X}) = \mathbb{X}(\frac{1}{\sigma})G^o(\sigma)$$

is a bijection.

PROOF. Suppose $\mathbb{X}, \mathbb{Y} \in L(G)$ and $S(\mathbb{X}) = S(\mathbb{Y})$. Set $a := \mathbb{X}(\frac{1}{\sigma})$ and $b := \mathbb{Y}(\frac{1}{\sigma})$. By Lemma 7.4, $a^{-1}b \in G^o(\sigma)$. Let $U \subseteq U_2$ be a compact symmetric neighborhood of 1, and let $\tau := min\{ord_U(a), \sigma\}$. Since for $i \in \{1, \ldots, \tau\}$, $a^i \in U^*$ and $(a^{-1}b)^i \in \mu$, we can use Lemma 7.3, so that for $i \in \{1, \ldots, \tau\}$, $(a.(a^{-1}b))^i$ is defined, nearstandard and infinitely close to a^i. That means for all $i \in \{1, \ldots, \tau\}$ such that $\frac{i}{\sigma} \in$ domain$(\mathbb{X}) \cap$ domain$(\mathbb{Y})$, we have $\mathbb{X}(\frac{i}{\sigma}) = \mathbb{Y}(\frac{i}{\sigma})$. Now notice that $\sigma = O(\tau)$ (otherwise $a^{ord_U(a)} \in \mu$), hence $\mathbb{X} = \mathbb{Y}$ (as they have equal reductions), i.e S is injective.

Next let $b \in G(\sigma)\backslash G^o(\sigma)$. By Lemma 5.3, one can consider the 1-ps X_b of G defined on $(-r_{b,\sigma,U}, r_{b,\sigma,U})$ by $X_b(t) = st(b^{\lfloor t\sigma \rfloor})$. Then let $b_1 := X_b(\frac{1}{\sigma}) \in \mu$. Recall that $\sigma = O(\tau)$ iff $\frac{\tau}{\sigma}$ is finite iff there is $r \in \mathbb{R}$, $|\frac{\tau}{\sigma}| < r$. We can thus pick $\tau \in \{1, \ldots, \sigma\}$ such that $\sigma = O(\tau)$ and $\frac{\tau}{\sigma} \in$ domain(X_b). Note that the latter means b_1^τ is defined and belongs to μ. Hence, for $i \in \{1, \ldots, \tau\}$, b^i and b_1^i are defined and $b^i \sim b_1^i$. Then by Lemma 7.4, $b^{-1}b_1 \in G^o(\tau) \subseteq G^o(\sigma)$ (see Lemma 4.2), hence $bG^o(\sigma) = b_1 G^o(\sigma)$, i.e. S is surjective. $\square$

Set

$$+_\sigma : \begin{array}{ccc} L(G) \times L(G) & \to & L(G) \\ (\mathbb{X}, \mathbb{Y}) & \mapsto & S^{-1}(S(\mathbb{X}).S(\mathbb{Y})) \end{array}$$

Corollary 7.7. *Let $\mathbb{X}, \mathbb{Y} \in L(G)$, and $t \in$ domain$(\mathbb{X} +_\sigma \mathbb{Y})$. If ν is such that $\frac{\nu}{\sigma} \sim t$, then*

$$(\mathbb{X} +_\sigma \mathbb{Y})(t) \sim \left[\mathbb{X}(\frac{1}{\sigma})\mathbb{Y}(\frac{1}{\sigma})\right]^\nu$$

Furthermore $L(G)$ equipped with the operation $+_\sigma$ is an abelian group.

Proof. By Lemma 5.7: $[X_a] = \mathbb{O}$ iff $a \in G^o(\sigma)$, hence $\mathbb{O}$ is the identity element of $L(G)$. Likewise, Lemma 5.7 gives the inverse element of $[X_a]$, namely $[X_{a^{-1}}]$. The associativity is clear:

$$\begin{aligned}(\mathbb{X} +_\sigma \mathbb{Y}) +_\sigma \mathbb{Z} &= S^{-1}(S(S^{-1}(S(\mathbb{X}).S(\mathbb{Y}))).S(\mathbb{Z})) \\ &= S^{-1}(S(\mathbb{X}).S(\mathbb{Y}).S(\mathbb{Z})) \\ &= \mathbb{X} +_\sigma (\mathbb{Y} +_\sigma \mathbb{Z})\end{aligned}$$

The fact that it is abelian is given by Theorem 7.5 item (3).

$\square$

3. Sketch of proof of local H5

Suppose G to be NSS. The first step is to equip $L(G)$ with a topology so as to make it a locally compact real vector space: a base of the topology is given by the sets

$$B(C, U) := \{\mathbb{X} \in L(G) : C \subseteq \text{domain}(\mathbb{X}) \text{ and } \mathbb{X}(C) \subseteq U\}$$

$C \subseteq \mathbb{R}$ compact and U open in G. (It is in fact the subspace topology of the 'compact-open' topology of the space $C(\mathbb{R}, G)$ consisting of continuous functions from $\mathbb{R}$ to G.

Having constructed local 1-ps from infinitesimals the powers of which grow neither too fast nor too slow allow to obtain that a neighborhood of the identity in G is ruled by local 1-ps (whereas if G is not NSS, a net of small connected subgroups is obtained instead): the local exponential map is a homeomorphism and $K = \{\mathbb{X}(1) : \mathbb{X} \in \mathcal{K}\}$ is a compact neighborhood of 1 in G. This shows that $L(G)$ is locally compact, hence, as a locally compact real vector space, is then a finite dimensional real vector space by a theorem of Riesz.

I.Goldbring then shows the local H5 for NSS local groups by using a local version of the Adjoint Representation Theorem, namely that there is a morphism of local groups $Ad : G|U_6 \to L(G)$, $g \mapsto Ad_g : L(G) \to L(G)$, $Ad_g([X_a]) = [X_{gag^{-1}}]$. The kernel is abelian and is a normal sublocal group of $G' = G|U_6$. Under other satisfied conditions involving the notion of a local quotient, he then uses a theorem of Kuranishi to show that a restriction of G is a local Lie group.

At the end he shows that locally Euclidean groups are NSS, first by showing that locally Euclidean local groups are NSCS, and then that locally connected NSCS local groups are NSS.

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